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Embedding partial totally symmetric quasigroups

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Abstract

This paper concerns the embedding problem for partial totally symmetric quasigroups. For all $n \geqslant 9$, it is shown that any partial totally symmetric quasigroup of order n can be embedded in a totally symmetric quasigroup of order v if v is even and $v \geqslant 2n+4$, and this is the best possible such inequality.

Keywords: Embedding; Totally symmetric; Partial latin square; Extended triple system

1. Introduction

A *partial binary operation* $*$ on a set V is a function $f: S \mapsto V$ where $S \subseteq V \times V$ and $f((a,b))$ is denoted by $a * b$. If $S = V \times V$, then we have a *binary operation*. A (*partial*) *quasigroup* is a pair $(V, *)$ where V is a set and $*$ is a (partial) binary operation on V such that for all $a, b \in V$, the equations $a * x = b$ and $x * a = b$ each have at most one solution for x in $(V, *)$. It follows that in any finite quasigroup, these equations each have exactly one solution. We shall be concerned only with finite quasigroups.

A quasigroup $(Q, \circ)$ is *idempotent* if it satisfies the identity $x \circ x = x$. A quasigroup $(Q, \circ)$ is *commutative* if it satisfies the identity $x \circ y = y \circ x$. A quasigroup $(Q, \circ)$ is *semi-symmetric* if it satisfies the identity $y \circ (x \circ y) = x$. Finally, a quasigroup $(Q, \circ)$ is *totally symmetric* if it satisfies both identities $x \circ y = y \circ x$ and $y \circ (x \circ y) = x$; that is, if it is both commutative and semi-symmetric. A *partial totally symmetric quasigroup* is a partial quasigroup such that if $x \circ y$ is defined, then $y \circ x$ and $(x \circ y) \circ x$ are defined and

$$x \circ y = y \circ x \quad \text{and} \quad (x \circ y) \circ x = y.$$

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This means that if $x \circ y$ is defined then we have $y \circ x = x \circ y$, $(x \circ y) \circ x = y$, $x \circ (x \circ y) = y$, $y \circ (x \circ y) = x$ and $(x \circ y) \circ y = x$.

A partial totally symmetric quasigroup $(U, \circ)$ is *embedded* in a partial totally symmetric quasigroup $(V, \star)$ if $U \subseteq V$ and whenever $a \circ b$ is defined, $a \star b$ is defined and $a \star b = a \circ b$. The problem of interest concerns embeddings of partial totally symmetric quasigroups in (complete) totally symmetric quasigroups.

Our main result (see Section 10) is that any partial totally symmetric quasigroup of order $n \geqslant 9$ has an embedding in a totally symmetric quasigroup of order v for all even $v \geqslant 2n + 4$. It is natural to ask whether this result is best possible. The answer is that it depends on how one defines 'best possible.' There are at least two reasonable ways. Firstly, we shall see in Section 4 that it is not possible to embed an arbitrary partial totally symmetric quasigroup of order n in a totally symmetric quasigroup of order v, if v is odd, or if $v < 2n + 4$. In this sense, obtaining embeddings for each even $v \geqslant 2n + 4$ is best possible. On the other hand, it may be true that every partial totally symmetric quasigroup of order n has an embedding in a totally symmetric quasigroup of order v for some $v < 2n + 4$, though the value of v will vary for different partial totally symmetric quasigroups.

Interest in the embedding problem for partial totally symmetric quasigroups is not new. In 1975, Chaffer et al. [6] proved that any partial totally symmetric quasigroup of order n can be embedded in a totally symmetric quasigroup of some finite order (exponential in n), and in 1976 Lindner and Cruse [10] proved that any partial totally symmetric quasigroup of order n has an embedding in a totally symmetric quasigroup of order v if $v \geqslant 6n$ and $v \equiv 0 \pmod 6$. Raines [12] and Raines and Rodger [13] made further progress by showing that any partial totally symmetric quasigroup of order n can be embedded in a totally symmetric quasigroup of order v if $v \geqslant 4n+4$ and v is even.

We now give a general outline of our overall embedding method. To do this, we use some terminology which is defined in Sections 2 and 3, as well as the equivalence of totally symmetric quasigroups and extended triple systems which is discussed in Section 3. Our method is an extension of the techniques developed in [3] to settle the embedding problem for partial Steiner triple systems.

Firstly, there are some partial extended triple systems, for example some having leaves with very high maximum degree and some with leaves having very few edges, that our general method does not cover. These are dealt with by other methods in Section 5. In particular, they are dealt with by using results on maximum packings of triples in complete graphs with the edges of complete subgraphs removed [4] and the result of Hoffman and Rodger [8] on embeddings of (complete) extended triple systems. The general method works as follows.

We wish to embed a given partial extended triple system (U_0', A_0') of order n' in an extended triple system of order v where v is even and $v \geqslant 2n' + 4$. We first let $n = \frac{v-4}{2}$, add $n - n'$ new elements and embed (U_0', A_0') in a maximal partial extended triple system (U_0, A_0) of order n by arbitrarily adding extended triples. We then add two further elements and construct a new partial extended triple system (U_1, A_1) of order $u = n + 2$. This is done in Section 6.

The partial extended triple system (U_1, A_1) is then embedded in a partial extended triple system of order $2u + 1 = 2n + 5 = v + 1$. We actually remove the proper extended triples from (U_1, A_1) before embedding it, ensure that the edges used in these triples remain in the leave, and then reinstate these triples later. This makes the description of the embedding simpler and allows us to use results from [3] more readily. The embedding of (U_1, A_1) in a partial extended triple system of order $2u + 1$ closely follows the partial Steiner triple system embedding method of [3], but needs to deal with some additional complications. Essentially, these arise because the degrees of the vertices in the leave of (U_1, A_1) need not all have the same parity (as is the case with leaves of partial Steiner triple systems). Thus we require some additional "repacking" techniques which we present in Section 7. The embedding of (U_1, A_1) in a partial extended triple system of order $2u + 1$ is split into two cases depending on whether the number of edges in the leave is large or small. These cases are covered in Sections 8 and 9, respectively.

Finally, the required extended triple system is obtained by deleting an element (so that the resulting order is $2u = 2n + 4 = v$) and adding a small number of proper extended triples. This is carried out in Lemma 10.1 and is based on observations concerning the sets of edges used in the proper extended triples of an extended triple system that were noted in [7]. The results are combined in Section 10 to prove the main result.

2. General notation and preliminaries

In this section we introduce some terminology and concepts which will be used throughout.

All graphs considered in this paper have at most one edge joining any given pair of distinct vertices, and may contain at most one loop on any vertex. We shall not refer to loops as edges; that is, an edge always joins distinct vertices. The set consisting of the edges of a graph G is denoted by $E(G)$, and the set consisting of the loops and edges of G is denoted by $E^+(G)$.

Let G be a graph and let $x \in V(G)$. The *degree* of x in G is the number of edges incident with x, and is denoted by $\deg_G(x)$. (Note that our convention is that loops do not contribute to the degree of a vertex.) The *neighbourhood* of x in G is the set of vertices distinct from x that are adjacent to x in G, and we denote this set by $\mathrm{Nbd}_G(x)$. The maximum degree of G is denoted by $\Delta(G)$, and the number of edges in G is denoted by $\epsilon(G)$.

The complete (loopless) graph with vertex set V is denoted by K_V, and the complete graph with vertex set V and with exactly one loop incident with each vertex is denoted by K_V^+. Let A and B be disjoint non-empty sets of orders a and b, respectively. The complete (loopless) bipartite graph $K_{a,b}$ is the graph with $V(K_{a,b}) = A \cup B$ and $xy \in E(K_{a,b})$ if and only if exactly one of x and y is in A, and the other is in B. An n-cycle in a graph G with vertices $a_1, a_2, \ldots, a_n \in V(G)$ and edges $a_1a_2, a_2a_3, \ldots, a_na_1 \in E(G)$ is denoted by $(a_1, a_2, \ldots, a_n)$.

For arbitrary edge-disjoint graphs G and H, the graph union of G and H will be denoted by $G \cup H$. That is, $V(G \cup H) = V(G) \cup V(H)$ and $E^+(G \cup H) = E^+(G) \cup E^+(H)$. Let F be a subgraph of G. The graph difference $G - F$ is the graph with $V(G - F) = V(G)$ and $E^+(G - F) = E^+(G) \setminus E^+(F)$.

For any graph G, the subgraph of G induced by a subset U of $V(G)$ will be denoted by $G(U)$. That is, $V(G(U)) = U$ and $xy \in E^+(G(U))$ if and only if $x, y \in U$ and $xy \in E^+(G)$. Suppose that A and B are disjoint subsets of $V(G)$. Then the graph $G(A, B)$ is defined by $V(G(A, B)) = A \cup B$ and $xy \in E^+(G(A, B))$ $(= E(G(A, B)))$ if and only if $xy \in E^+(G)$ and exactly one of x and y is in A, and the other is in B. So if $U \cup W$ is a partition of $V(G)$ then $E^+(G)$ is the disjoint union of $E^+(G(U))$, $E^+(G(W))$ and $E^+(G(U, W))$.

We define a graph H of order v to be *useable* if it satisfies the following four conditions.

- $\epsilon(H) \equiv 0 \pmod 3$.
- $\Delta(H) \leqslant v - 5$.
- $\epsilon(H) \leqslant \frac{1}{2}(v(v - 5) - 2)$.

- Either
  - $\deg_H(x) \equiv v - 1 \pmod 2$ for all $x \in V(H)$; or
  - there are precisely two vertices $x_1, x_2 \in V(H)$ with $\deg_H(x_1) \equiv \deg_H(x_2) \equiv v \pmod 2$, and $\deg_H(x) \equiv v - 1 \pmod 2$ for all $x \in V(H) \setminus \{x_1, x_2\}$.

Useable graphs play an important role in the constructions described in later sections.

An *edge colouring* of a graph G is an assignment of colours to its edges. An edge colouring is *proper* if adjacent edges are assigned distinct colours. The minimum number of colours needed for a proper edge colouring of G is denoted by $\chi'(G)$. Vizing [16] showed that for any graph G, $\chi'(G) \leqslant \Delta(G) + 1$. An edge colouring with colours $c_1, c_2, \dots, c_t$ is *equalised* if $|\gamma[c_i] - \gamma[c_j]| \leqslant 1$ for all $i, j \in \{1, 2, \dots, t\}$, where $\gamma[c_i]$ denotes the number of edges of G coloured c_i. It is well known that any graph having a proper edge colouring with t colours, has an equalised proper edge colouring with t colours (see [11,17]).

Let H be a graph, let γ be an edge colouring of H with colour set C, and let a and b be two distinct colours in C. An *alternating (a,b)-path* is a path in H with edges alternately coloured a and b. Suppose γ is a proper edge colouring and $P = x_1, x_2, \dots, x_t$ is a maximal alternating (a,b)-path which is not an alternating cycle with an edge removed. If we interchange the colours a and b on the edges of P, then the result is a new proper edge colouring of H. We call this recolouring process *switching along the alternating (a,b)-path from x_1 to x_t*. Let x_1 be a vertex in a properly edge coloured graph H, and suppose some colour a is not assigned to any of the edges incident with x_1. Then for any colour $b \neq a$ there is a maximal alternating (a,b)-path $x_1, x_2, \dots, x_t$ in H (this path consists of the single vertex $x_1 = x_t$ if b is not assigned to any edge incident with x_1). We call $x_1, x_2, \dots, x_t$ the alternating (a,b)-path *emanating* from x_1 and *terminating* at x_t.

3. Partial extended triple systems

In this section we outline the connection between totally symmetric quasigroups and certain triple systems. We also give some definitions and notation, and describe some simple properties of the systems which will be used in the remainder of the paper.

A (*partial*) *Steiner triple system of order v* is a pair (V, B), where V is a set with v elements and B is a set of triples chosen from V such that every pair of distinct elements of V occurs in (at most) exactly one triple of B.

One may consider pairs of distinct elements of V as corresponding to edges of K_V. A pair which occurs in a triple of B will be called a *used edge* and the spanning subgraph of K_V induced by the used edges will be called the *underlying graph* of the system. The *leave* is the spanning subgraph of K_V which has edge set consisting of the unused edges (that is, the leave is the complement of the underlying graph). If $U \subseteq V$, then we say that a triple of B is a *U-type i triple*, $i \in \{0, 1, 2, 3\}$ if it has exactly i elements in U.

For any partial Steiner triple system (V, B), the triples in B induce a proper edge colouring, with colour set V, of the underlying graph of (V, B). The colouring is given by $\gamma(xy) = z$ if and only if $\{x, y, z\} \in B$. We will assume that the underlying graph of any partial Steiner triple system is coloured according to this induced proper edge colouring.

There is a well-known equivalence between Steiner triple systems and idempotent totally symmetric quasigroups (see [2]). Indeed, such quasigroups are often called Steiner quasigroups. A totally symmetric quasigroup, which is not necessarily idempotent, is equivalent to an *extended triple system* [9] which we now define.

An *extended triple* is a triple of the form $\{a, b, c\}$ where a, b and c are chosen from a set V and are not necessarily distinct. There are three possible types of extended triples:

- A *loop* is an extended triple of the form $\{a, a, a\}$ for some $a \in V$.
- A *lollipop* is an extended triple of the form $\{a, a, b\}$ for some distinct $a, b \in V$.
- A *normal triple* is an extended triple of the form $\{a, b, c\}$ for some distinct $a, b, c \in V$.

We shall refer to loops and lollipops as *proper extended triples*.

A (*partial*) *extended triple system of order* v is a pair (V, B), where V is a set with v elements and B is a set of extended triples chosen from V, such that every pair of elements of V, distinct or otherwise, occurs in (at most) exactly one extended triple in B.

One may consider pairs of (not necessarily distinct) elements of V as corresponding to edges and loops of K_V^+. Similarly to the case of partial Steiner triple systems, a pair which occurs in an extended triple of B will be called a *used edge* or *used loop*, and the spanning subgraph of K_V^+ induced by the used edges and used loops will be called the *underlying graph* of the system. The *leave* is the spanning subgraph of K_V^+ which has edge set consisting of the unused edges and unused loops.

Note that (U, A) is a partial Steiner triple system if and only if (U, A) is a partial extended triple system consisting entirely of normal triples. However, the leave of (U, A) depends on whether it is considered as a partial Steiner triple system or as a partial extended triple system. The underlying graph in each case is the same, but if (U, A) is specified to be a partial extended triple system then its leave is defined by the unused edges and unused loops of K_U^+, whereas if (U, A) is specified to be a partial Steiner triple system then its leave is defined by the unused edges of K_U. In this situation, it will always be made clear whether (U, A) is considered a partial Steiner triple system or a partial extended triple system.

Let (U, A) be a partial extended (Steiner) triple system and let $x \in U$. The partial extended (Steiner) triple system $(U, A) \setminus x$ is defined to be the partial extended (Steiner) triple system obtained from (U, A) by removing x, and all the triples in A containing x. That is,

$$(U, A) \setminus x = \bigl(U \setminus \{x\}, A \setminus \{T \in A\colon\ x \in T\}\bigr).$$

The definition of embedding extends naturally from quasigroups to triple systems. A partial extended (Steiner) triple system (U, A) is *embedded* in a partial extended (Steiner) triple system (V, B) if $U \subseteq V$ and $A \subseteq B$. A partial extended (Steiner) triple system (U, A) is *maximal* if there is no partial extended (Steiner) triple system (U, A') with A a proper subset of A'. Note that the leave L of a maximal partial extended triple system contains no loops, and so there is no distinction between $E(L)$ and $E^+(L)$. Henceforth, we shall phrase our discussion in terms of extended triple systems rather than totally symmetric quasigroups.

Suppose that a partial extended triple system (U, A) of order u is embedded in an extended triple system (V, B) of order v. It is easy to see that for each element $x \in V$, the number of lollipops in B containing x is congruent to $v - 1 \pmod 2$. Thus, for each $y \in U$, the parity of the number of lollipops containing y is of interest. We say that y is *popped in* A if it is in an odd number of lollipops in A and *unpopped in* A if it is in an even number of lollipops in A. We let $\beta((U, A))$ denote the number of elements in U which are unpopped in A. Note that $\beta((U, A)) \equiv u \pmod 2$ and that every element of an extended triple system of even order is popped.

Let (V, B) be an extended triple system of order v. There are v used loops and $\binom{v}{2}$ used edges in (V, B). It is clear that B contains precisely v proper extended triples. Suppose that l of these proper extended triples are lollipops, and that B contains precisely t normal triples. Then $\binom{v}{2} = l + 3t$ and so $l \equiv \binom{v}{2} \pmod 3$.

4. Necessary conditions

As mentioned in the introduction, it is not possible to embed an arbitrary partial totally symmetric quasigroup of order n in a totally symmetric quasigroup of order v when v is odd or when $v < 2n + 4$. In this section we prove this fact.

Lemma 4.1. *For all $n \geqslant 2$, there exists a partial extended triple system of order n which cannot be embedded in any extended triple system of odd order.*

Proof. Let U be a set with $|U| = n$, let $a, b \in U$ with $a \neq b$, and let $\{\{a, a, a\}, \{a, b, b\}\} \subseteq A$. We claim that (U, A) cannot be embedded in any extended triple system of odd order. Suppose otherwise, and let (V, B) be an extended triple system of odd order v in which (U, A) is embedded.

Since there are v proper extended triples in B, and since $\{a, a, a\}$ and $\{a, b, b\}$ are two of them, it follows that there is some $x \in V$ which occurs in no lollipop of the form $\{x, y, y\}$ and in no loop of the form $\{x, x, x\}$. However, since the pair $\{x, x\}$ occurs in some triple of B, say $\{z, x, x\}$ (where $z \neq x$), x is popped in B. This is a contradiction, since v is odd implies that all the elements of V are unpopped in B. $\square$

Lemma 4.2. *Let $n \geqslant 6$, with $n \equiv 0, 1 \pmod 3$. There exists a partial extended triple system of order n which cannot be embedded in any extended triple system of even order less than $2n + 4$.*

Proof. Let $U = \{x_1, x_2, \ldots, x_n\}$. If $n \equiv 0 \pmod 3$, then we let $A = \{\{x_1, x_1, x_2\}, \{x_2, x_2, x_3\}, \{x_3, x_3, x_4\}, \{x_4, x_4, x_1\}\} \cup \{\{x_i, x_i, x_i\}\colon 5 \leqslant i \leqslant n\}$. If $n \equiv 1 \pmod 3$, then we let $A = \{\{x_1, x_1, x_2\}, \{x_2, x_2, x_3\}, \{x_3, x_3, x_4\}, \{x_4, x_4, x_5\}, \{x_5, x_5, x_1\}\} \cup \{\{x_i, x_i, x_i\}\colon 6 \leqslant i \leqslant n\}$. We claim that (U, A) cannot be embedded in any extended triple system of even order less than $2n + 4$. Suppose otherwise, and let (V, B) be an extended triple system of even order v, where $v \leqslant 2n + 2$, in which (U, A) is embedded. Let the elements of $V \setminus U$ be $y_1, y_2, \ldots, y_{v-n}$.

Each element of V is popped in B (since v is even), and $x_1, x_2, \ldots, x_n$ are unpopped in A. Thus, $x_1, x_2, \ldots, x_n$ each occur in at least one lollipop in $B \setminus A$. The loops $x_i x_i$ for $i = 1, 2, \ldots, n$ are used in extended triples in A. Thus, without loss of generality B contains the n lollipops $\{x_1, y_1, y_1\}, \{x_2, y_2, y_2\}, \ldots, \{x_n, y_n, y_n\}$. Hence $|V \setminus U| \geqslant n$ and so $v \geqslant 2n$. We now consider separately the two cases $v = 2n$ and $v = 2n + 2$. In each, we let l denote the number of lollipops in B (recall from Section 3 that $l \equiv \binom{v}{2} \pmod 3$).

Suppose that $v = 2n$. Then we have already accounted for the $2n$ proper extended triples in B (the n extended triples of A and the n lollipops $\{x_1, y_1, y_1\}, \{x_2, y_2, y_2\}, \ldots, \{x_n, y_n, y_n\}$). If $n \equiv 0 \pmod 3$ we have $l = n + 4$, and if $n \equiv 1 \pmod 3$ we have $l = n + 5$. But then $l \not\equiv \binom{2n}{2} \pmod 3$ in either case, and so we have a contradiction.

Now suppose that $v = 2n + 2$. In this case B contains two proper extended triples in addition to the n extended triples of A and the n lollipops $\{x_1, y_1, y_1\}, \{x_2, y_2, y_2\}, \ldots, \{x_n, y_n, y_n\}$. But y_{n+1} and y_{n+2} each appear in an odd number of lollipops (and so, at least one lollipop) in B. Hence, there are either one or two lollipops in B containing y_{n+1} or y_{n+2}. From these facts, we obtain the following bounds on l. If $n \equiv 0 \pmod 3$ we have $n + 5 \leqslant l \leqslant n + 6$, and if

$n \equiv 1 \pmod 3$ we have $n+6 \leqslant l \leqslant n+7$. But then $l \not\equiv \binom{2n+2}{2} \pmod 3$ in either case, and so we have a contradiction. □

Lemma 4.3. *Let $n \geqslant 8$, with $n \equiv 2 \pmod 3$. There exists a partial extended triple system of order n which cannot be embedded in any extended triple system of odd order, and cannot be embedded in any extended triple system of even order less than $2n+2$.*

Proof. Let $U = \{x_1, x_2, \ldots, x_n\}$ and let $A = \{\{x_1, x_1, x_1\}, \{x_1, x_2, x_2\}, \{x_3, x_3, x_4\}, \{x_4, x_4, x_5\}, \{x_5, x_5, x_6\}, \{x_6, x_6, x_7\}, \{x_7, x_7, x_3\}\} \cup \{\{x_i, x_i, x_i\}\colon\ 8 \leqslant i \leqslant n\}$. Since $\{\{x_1, x_1, x_1\}, \{x_1, x_2, x_2\}\} \subseteq A$, Lemma 4.1 guarantees that (U, A) cannot be embedded in any extended triple system of odd order. We claim that (U, A) cannot be embedded in any extended triple system of even order less than $2n+2$. Suppose otherwise, and let (V, B) be an extended triple system of even order v, where $v \leqslant 2n$, in which (U, A) is embedded. Let the elements of $V \setminus U$ be $y_1, y_2, \ldots, y_{v-n}$.

Each element of V is popped in B (since v is even), and $x_3, x_4, \ldots, x_n$ are unpopped in A. Thus, $x_3, x_4, \ldots, x_n$ each occur in at least one lollipop in $B \setminus A$. The loops $x_i x_i$ for $i = 1, 2, \ldots, n$ are used in extended triples in A. Thus, without loss of generality B contains the n lollipops $\{x_3, y_1, y_1\}, \{x_4, y_2, y_2\}, \ldots, \{x_n, y_{n-2}, y_{n-2}\}$. Hence $|V \setminus U| \geqslant n-2$ and so $v \geqslant 2n-2$. We now consider separately the two cases $v = 2n-2$ and $v = 2n$. In each, we let l denote the number of lollipops in B.

Suppose that $v = 2n-2$. Then we have already accounted for the $2n-2$ proper extended triples in B (the n extended triples of A and the $n-2$ lollipops $\{x_3, y_1, y_1\}, \{x_4, y_2, y_2\}, \ldots, \{x_n, y_{n-2}, y_{n-2}\}$). Thus, we have $l = n+4$. But then $l \not\equiv \binom{2n-2}{2} \pmod 3$, and so we have a contradiction.

Now suppose that $v = 2n$. In this case B contains two proper extended triples in addition to the n extended triples of A and the $n-2$ lollipops $\{x_3, y_1, y_1\}, \{x_4, y_2, y_2\}, \ldots, \{x_n, y_{n-2}, y_{n-2}\}$. But y_{n-1} and y_n each appear in an odd number of lollipops (and so, at least one lollipop) in B. Hence, there are either one or two lollipops in B containing y_{n-1} or y_n. From these facts, we have $n+5 \leqslant l \leqslant n+6$. But then $l \not\equiv \binom{2n}{2} \pmod 3$ and we obtain a contradiction. □

Lemma 4.4. *Let $n \geqslant 5$, with $n \equiv 2 \pmod 3$. There exists a partial extended triple system of order n which cannot be embedded in any extended triple system of order $2n+2$.*

Proof. Let $U = \{x_1, x_2, \ldots, x_n\}$ and let $A = \{\{x_1, x_1, x_2\}, \{x_2, x_2, x_3\}, \{x_3, x_3, x_4\}, \{x_4, x_4, x_1\}\} \cup \{\{x_i, x_i, x_i\}\colon\ 5 \leqslant i \leqslant n\}$. We claim that (U, A) cannot be embedded in any extended triple system of order $2n+2$. Suppose otherwise, and let (V, B) be an extended triple system of order $2n+2$ in which (U, A) is embedded. Let the elements of $V \setminus U$ be $y_1, y_2, \ldots, y_{n+2}$.

Each element of V is popped in B (since n is even), and $x_1, x_2, \ldots, x_n$ are unpopped in A. Thus, $x_1, x_2, \ldots, x_n$ each occur in at least one lollipop in $B \setminus A$. The loops $x_i x_i$ for $i = 1, 2, \ldots, n$ are used in extended triples in A. Hence, without loss of generality B contains the n lollipops $\{x_1, y_1, y_1\}, \{x_2, y_2, y_2\}, \ldots, \{x_n, y_n, y_n\}$. The n proper extended triples of A (four of which are lollipops), together with these n lollipops account for all except two of the $2n+2$ proper extended triples of B. The elements y_{n+1} and y_{n+2} are popped in B (and so must each appear in at least one lollipop in B). Hence, there are either one or two lollipops in B which contain y_{n+1} or y_{n+2}. From these facts, we see that the number l of lollipops in B is either $n+5$ or $n+6$. But then $l \not\equiv \binom{2n+2}{2} \pmod 3$ so we have a contradiction. □

The preceding four lemmas combine to yield the following theorem.

Theorem 4.5. *Let $n \geqslant 6$. For each $v < 2n + 4$ and for each odd v, there exists a partial extended triple system of order n which cannot be embedded in an extended triple system of order v.*

5. Special cases

In this section, we deal with four special cases which are not covered by our general method. To prove them, we use the following result from [4] on the existence of certain partial Steiner triple systems, and the result of Hoffman and Rodger [8] on embeddings of (complete) extended triple systems.

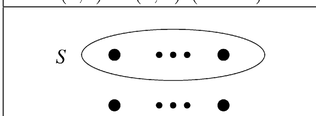
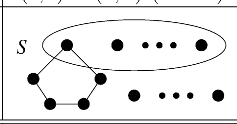
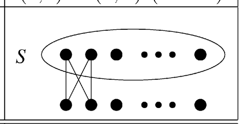
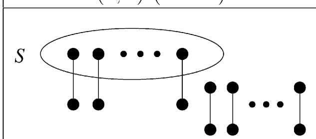
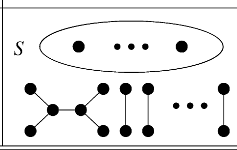
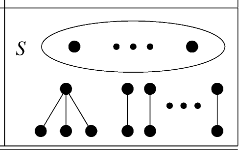
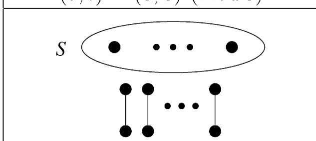
Lemma 5.1. [4] *Let S and T be sets with $S \subseteq T$, $|S| = s$ and $|T| = t$ such that $t \geqslant 2s + 1$. For the indicated congruence classes of s and t modulo 6, there exists a partial Steiner triple system (T, B) with a leave having edge set consisting of the edges of K_S, and the edges indicated in Table 1.*

We will also require the following simple lemma, based on observations that were noted in [7], concerning the sets of edges used in the proper extended triples of an extended triple system.

Lemma 5.2. *Let U and W be disjoint sets, let $(U \cup W, A)$ be a partial extended triple system of order v and let its leave be L. Suppose that L consists of a union of vertex-disjoint trees, each with at most one vertex in U, and a loop on each vertex of W. Then $(U \cup W, A)$ can be embedded in an extended triple system $(U \cup W, B)$ of order v.*

Proof. Consider an arbitrary tree T, let x be any vertex of T, and direct all edges of T away from x (so x is the *root* of T). Then let T^* denote the graph obtained from T by adding a loop to each vertex of T except x. It is clear that T^* is the underlying graph of a set of lollipops; for each directed edge (a, b), we have the lollipop $\{a, b, b\}$.

Table 1

$(s,t) \equiv (1,1) \pmod 6$	$(s,t) \equiv (5,3) \pmod 6$	$(s,t) \equiv (3,5) \pmod 6$
		
$(s,t) \equiv (1,2), (3,0), (5,4) \pmod 6$	$(s,t) \equiv (4,2) \pmod 6$	$(s,t) \equiv (2,4) \pmod 6$
		
$(s,t) \equiv (0,0) \pmod 6$		
		

Thus, it is clear that there exists a set X of loops and lollipops such that the edges and loops used in X are precisely the edges and loops of L. (For any tree T containing a vertex x in U, choose x to be the root, so that all the vertices of T other than the root have loops on them in L.) Then $(U \cup W, A \cup X)$ is the required extended triple system of order v. □

Lemma 5.3. *Let $n \geqslant 2$. Let (U_0, A_0) be a maximal partial extended triple system of order n, let L_0 be its leave and suppose that $\Delta(L_0) = n-1$. Then (U_0, A_0) can be embedded in an extended triple system of order $2n+4$.*

Proof. Since (U_0, A_0) is maximal and $\Delta(L_0) = n-1$, it is clear that $L_0 = K_{1,n-1}$. Let $a \in U_0$ be the vertex satisfying $\deg_{L_0}(a) = n-1$ and let $S = U_0 \setminus \{a\}$. Then $(U_0, A_0) \setminus a = (S, A_0 \setminus \{\{a,a,a\}\})$ is an extended triple system of order $n-1$. To embed (U_0, A_0) in an extended triple system of order $2n+4$, we will construct an embedding of $(U_0, A_0) \setminus a$ in an extended triple system of order $2n+4$ containing a loop $\{b,b,b\}$ for some $b \notin S$. Relabelling b as a then yields the required embedding of (U_0, A_0). There are four cases to consider, depending on the congruence of n (mod 6).

Case 1. n is even.

For n even, we have $(|S|, 2n+4) \pmod 6 \in \{(5,4), (1,2), (3,0)\}$. Thus, there exists a partial Steiner triple system of order $2n+4$ with a leave consisting of a complete subgraph on S and $n+2$ additional edges as depicted in Lemma 5.1. Clearly this partial Steiner triple system can be embedded in an extended triple system of order $2n+4$ by the addition of the triples of $A_0 \setminus \{\{a,a,a\}\}$, $n+2$ lollipops and three loops. Let one of these loops be $\{a,a,a\}$.

Case 2. $n \equiv 1 \pmod 6$.

For $n \equiv 1 \pmod 6$, we have $(|S|, 2n+4) \equiv (0,0) \pmod 6$. Thus, there exists a partial Steiner triple system of order $2n+4$ with a leave consisting of a complete subgraph on S and $\frac{n+5}{2}$ additional edges as indicated in Lemma 5.1. Clearly this partial Steiner triple system can be embedded in an extended triple system of order $2n+4$ by the addition of the triples of $A_0 \setminus \{\{a,a,a\}\}$, $\frac{n+5}{2}$ lollipops and $\frac{n+5}{2}$ loops. Let one of these loops be $\{a,a,a\}$.

Case 3. $n \equiv 3 \pmod 6$.

For $n \equiv 3 \pmod 6$, we have $(|S|, 2n+4) \equiv (2,4) \pmod 6$. Thus, there exists a partial Steiner triple system of order $2n+4$ with a leave consisting of a complete subgraph on S and $\frac{n+1}{2}+3$ additional edges as depicted in Lemma 5.1. Clearly this partial Steiner triple system can be embedded in an extended triple system of order $2n+4$ by the addition of the triples of $A_0 \setminus \{\{a,a,a\}\}$, $\frac{n+1}{2}+3$ lollipops and $\frac{n+1}{2}+1$ loops. Let one of these loops be $\{a,a,a\}$.

Case 4. $n \equiv 5 \pmod 6$.

For $n \equiv 5 \pmod 6$, we have $(|S|, 2n+4) \equiv (4,2) \pmod 6$. Thus, there exists a partial Steiner triple system of order $2n+4$ with a leave consisting of a complete subgraph on S and $\frac{n-1}{2}+5$ additional edges as depicted in Lemma 5.1. Clearly this partial Steiner triple system can be embedded in an extended triple system of order $2n+4$ by the addition of the triples of $A_0 \setminus \{\{a,a,a\}\}$, $\frac{n-1}{2}+5$ lollipops and $\frac{n-1}{2}+1$ loops. Let one of these loops be $\{a,a,a\}$. □

Lemma 5.4. *Let $n \geqslant 7$. Let (U_0, A_0) be a maximal partial extended triple system of order n, let L_0 be its leave and suppose that $\beta((U_0, A_0)) = n - 2$ and $\epsilon(L_0) \in \{1, 2\}$. Then (U_0, A_0) can be embedded in an extended triple system of order $2n + 4$.*

Proof. It is easy to see that $\beta((U_0, A_0)) = n - 2$ and $\epsilon(L_0) \in \{1, 2\}$ imply that n is odd (since any isolated vertex in the leave of a partial extended triple system of even order is necessarily popped). Furthermore, if $\epsilon(L_0) = 2$ then L_0 consists of a path of length two and $n - 3$ isolated vertices.

We shall construct a partial Steiner triple system (X, C) of order $2n + 4$ with $U_0 \subset X$ and with a leave G satisfying the following two properties:

- $G(U_0)$ has edge set consisting of the used edges of (U_0, A_0).
- $G - G(U_0)$ consists of vertex disjoint trees, each with at most one vertex in U_0.

Then it is clear that the partial extended triple system $(X, C \cup A_0)$ will be an embedding of (U_0, A_0) satisfying the conditions of Lemma 5.2. Applying Lemma 5.2 will thus yield the required embedding of (U_0, A_0).

We now show that we can construct the partial Steiner triple system (X, C). Let V be a set with $|V| = 2n + 5$ and $U_0 \subseteq V$. There are three cases to consider, depending on the congruence of $n \pmod 6$.

Case 1. $n \equiv 1 \pmod 6$.

For $n \equiv 1 \pmod 6$, we have $(|U_0|, 2n+5) \equiv (1, 1) \pmod 6$. Thus, there exists a partial Steiner triple system (V, B_0) of order $2n + 5$ with a leave consisting of a complete subgraph on U_0 and $n + 5$ isolated vertices (see Lemma 5.1). It is clear that each vertex in $V \setminus U_0$ is in precisely two U_0-type 0 triples and precisely n U_0-type 1 triples in B_0.

We first deal with the case where $\epsilon(L_0) = 1$. Let $\{a, b, c\}$ be a U_0-type 0 triple in B_0 and let $d \in V \setminus U_0$ such that d is not in any U_0-type 0 triple with a or b (such a vertex exists since $n \geqslant 7$). Then the edges ad and bd are used in two distinct U_0-type 1 triples in B_0, say $\{p, a, d\}$ and $\{q, b, d\}$, respectively, for some distinct $p, q \in U_0$.

Let (X, B_1) denote the partial Steiner triple system $(V, B_0) \setminus c$ of order $2n + 4$, and denote its leave by G_1. Clearly, G_1 consists of a complete subgraph on U_0 and $n + 2$ additional edges of the form xy where $\{c, x, y\} \in B_0$. In particular, $ab \in E(G_1)$ since $\{a, b, c\} \in B_0$.

Let $C = (B_1 \setminus \{\{p, a, d\}, \{q, b, d\}\}) \cup \{\{p, q, d\}, \{a, b, d\}\}$. Then the leave G of the partial Steiner triple system (X, C) of order $2n + 4$ has edge set consisting of $\{xy\colon x, y \in U_0\} \setminus \{pq\}$ and the $n + 3$ additional edges shown in Fig. 1 Case 1(i) below.

We now consider the case where $\epsilon(L_0) = 2$. Let $\{q, a, b\} \in B_0$ for some $q \in U_0$ and some $a, b \in V \setminus U_0$. Since $n \geqslant 7$, it is clear that there exist triples $\{a, p, c\}, \{b, r, c\} \in B_0$ for some $c \in V \setminus U_0$ and some distinct $p, r \in U_0 \setminus \{q\}$.

Let (X, B_1) denote the partial Steiner triple system $(V, B_0) \setminus c$ of order $2n + 4$, and denote its leave by G_1. Clearly, G_1 consists of a complete subgraph on U_0 and $n + 2$ additional edges of the form xy where $\{c, x, y\} \in B_0$. In particular, $ap, br \in E(G_1)$ since $\{a, p, c\}, \{b, r, c\} \in B_0$.

Let $C = (B_1 \setminus \{\{q, a, b\}\}) \cup \{\{p, q, a\}, \{q, r, b\}\}$. Then the leave G of the partial Steiner triple system (X, C) has edge set $\{xy\colon x, y \in U_0\} \setminus \{pq, qr\}$ and the $n + 1$ additional edges shown in Fig. 1 Case 1(ii).

Case 2. $n \equiv 3 \pmod 6$.

For $n \equiv 3 \pmod 6$, we have $(|U_0|, 2n+5) \equiv (3, 5) \pmod 6$. Thus, there exists a partial Steiner triple system (V, B_0) of order $2n + 5$ with a leave consisting of a complete subgraph on U_0, the additional edges of a 4-cycle (p, a, q, b) for some $p, q \in U_0$ and some $a, b \in V \setminus U_0$, and $n + 3$ isolated vertices (see Lemma 5.1). Let $\{r, b, c\}$ be a U_0-type 1 triple in B_0 where $r \in U_0 \setminus \{p, q\}$ and $c \in V \setminus (U_0 \cup \{a, b\})$ (such a triple exists since $n \geqslant 7$).

We first treat the case where $\epsilon(L_0) = 1$. Let (X, B_1) denote the partial Steiner triple system $(V, B_0) \setminus a$ of order $2n + 4$, and denote its leave by G_1. Clearly, G_1 consists of a complete subgraph on U_0, the two edges pb and qb, and $n + 1$ additional edges of the form xy where $\{a, x, y\} \in B_0$.

Let $C = B_1 \cup \{\{p, q, b\}\}$. Then the leave G of the partial Steiner triple system (X, C) has edge set $\{xy\colon x, y \in U_0\} \setminus \{pq\}$ and the $n + 1$ additional edges shown in Fig. 1 Case 2(i).

We now deal with the case where $\epsilon(L_0) = 2$. Let (X, B_1) denote the partial Steiner triple system $(V, B_0) \setminus c$ of order $2n+4$, and denote its leave by G_1. Clearly, G_1 consists of a complete subgraph on U_0, the edges of the 4-cycle (p, a, q, b), and $n + 2$ edges of the form xy where $\{c, x, y\} \in B_0$. In particular, $rb \in E(G_1)$ since $\{r, b, c\} \in B_0$.

Let $C = B_1 \cup \{\{p, q, a\}, \{q, r, b\}\}$. Then the leave G of the partial Steiner triple system (X, C) has edge set $\{xy\colon x, y \in U_0\} \setminus \{pq, qr\}$ and the $n+2$ additional edges shown in Fig. 1 Case 2(ii).

Case 3. $n \equiv 5 \pmod 6$.

For $n \equiv 5 \pmod 6$, we have $(|U_0|, 2n + 5) \equiv (5, 3) \pmod 6$. Thus, there exists a partial Steiner triple system (V, B_0) of order $2n + 5$ with a leave consisting of a complete subgraph on U_0, a 5-cycle (q, a, b, c, d), where $q \in U_0$ and $a, b, c, d \in V \setminus U_0$, and $n + 1$ isolated vertices (see Lemma 5.1). Furthermore, an inspection of the construction of (V, B_0) given in [4] reveals that $\{p, a, c\}, \{r, d, e\}, \{e, f, c\} \in B_0$ for some distinct $p, r \in U_0 \setminus \{q\}$ and some distinct $e, f \in V \setminus (U_0 \cup \{a, b, c, d\})$.

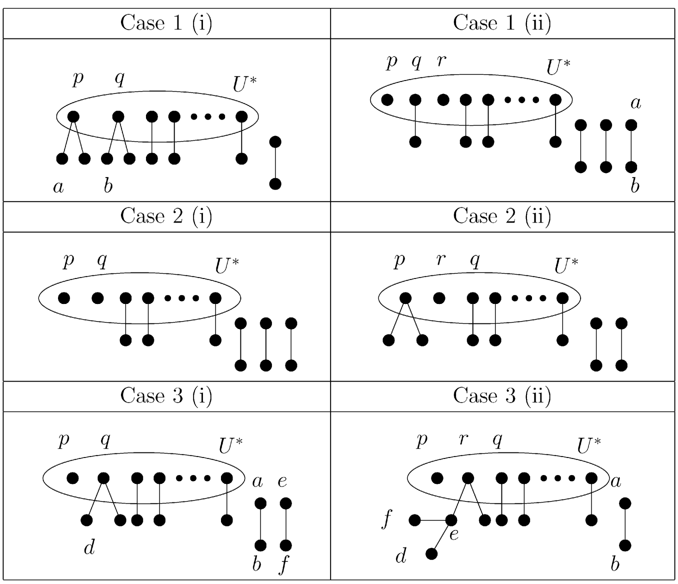


Fig. 1.

Deleting the vertex c (and all triples containing it) results in a partial Steiner triple system (X, B_1) of order $2n+4$ with a leave consisting of a complete subgraph on U_0, the edges qd, qa, and ab, and $n+1$ additional edges of the form xy where $\{c, x, y\} \in B_0$. In particular, pa is an edge in the leave, since $\{p, a, c\} \in B_0$.

If $\epsilon(L_0) = 1$ then we let $C = B_1 \cup \{\{p, q, a\}\}$. If $\epsilon(L_0) = 2$ then we let $C = (B_1 \setminus \{\{r, d, e\}\}) \cup \{\{p, q, a\}, \{q, r, d\}\}$. We denote the leave of the partial Steiner triple system (X, C) by G.

It is clear that if $\epsilon(L_0) = 1$ or $\epsilon(L_0) = 2$ then $E(G(U_0)) = \{xy\colon\ x, y \in U_0\} \setminus \{pq\}$ or $E(G(U_0)) = \{xy\colon\ x, y \in U_0\} \setminus \{pq, qr\}$, respectively. The edges of $G - G(U_0)$ in both cases are shown Fig. 1 Cases 3(i) or (ii), respectively. □

Lemma 5.5. *Let $n \geqslant 4$. Let (U_0, A_0) be a maximal partial extended triple system of order n, let L_0 be its leave and suppose that $\beta((U_0, A_0)) = n - 2$ and $L_0 = K_{1,n-2} \cup K_1$. Then (U_0, A_0) can be embedded in an extended triple system of order $2n + 4$.*

Proof. It is clear that $\beta((U_0, A_0)) = n - 2$ and $L_0 = K_{1,n-2} \cup K_1$ together imply that n is even. Let $a, b \in U_0$ be the vertices satisfying $\deg_{L_0}(a) = n - 2$ and $\deg_{L_0}(b) = 0$, and let $S = U_0 \setminus \{a\}$.

Since n is even, we have $(|S|, 2n + 4) \pmod 6 \in \{(5, 4), (1, 2), (3, 0)\}$. Thus, there exists a partial Steiner triple system of order $2n + 4$ with a leave L consisting of a complete subgraph on S and $n + 2$ additional edges as depicted in Lemma 5.1. We label with a the unique neighbour in L of b which is not in S. It is easy to see that this partial Steiner triple system of order $2n + 4$ can be embedded in an extended triple system of order $2n + 4$ by the addition of the triples of A_0, $n + 1$ lollipops and three loops. Thus, (U_0, A_0) is embedded in the resulting extended triple system of order $2n + 4$. □

The special case where L_0 is empty is covered in [8].

Theorem 5.6. [8] *An extended triple system of order n can be embedded in an extended triple system of order $v > n$ if and only if $v \geqslant 2n$, v is even if n is, and $(n, v) \neq (6k + 5, 12k + 12)$.*

6. A preliminary embedding

In this section, we prove three technical lemmas which are the first step in our general embedding method. In each, we embed a maximal partial extended triple system of order $u - 2$ in a partial extended triple system of order u. Lemma 6.1 deals with systems having all elements unpopped, Lemma 6.2 considers systems with precisely two popped elements, and Lemma 6.3 treats systems having at least four popped elements.

Lemma 6.1. *Let $u \geqslant 11$. Let (U_0, A_0) be a maximal partial extended triple system of order $u - 2$, and let its leave be L_0. If $\beta((U_0, A_0)) = u - 2$, L_0 is non-empty and $\Delta(L_0) < u - 3$, then (U_0, A_0) can be embedded in a partial extended triple system of order u, with a useable leave which contains no loops.*

Proof. First note that $\beta((U_0, A_0)) = u - 2$ implies that $\deg_{L_0}(x) \equiv u - 3 \pmod 2$ for all $x \in U_0$. This parity condition, together with the assumption that $\Delta(L_0) < u - 3$, implies that $\Delta(L_0) \leqslant u - 5$. Furthermore, since $u \geqslant 11$, L_0 is non-empty, (U_0, A_0) is maximal, and $\deg_{L_0}(x) \equiv u - 3 \pmod 2$ for all $x \in U_0$, we also have that $\epsilon(L_0) \geqslant 4$.

Let $p, q \notin U_0$. Consider the partial extended triple system $(U_0 \cup \{p, q\}, A_0)$, and denote its leave by G. Note that for all $x \in U_0$, $\deg_G(x) \equiv u - 1 \pmod 2$ and $\deg_G(x) > u - 5$ if and only if $\deg_{L_0}(x) = u - 5$.

We define A_0' by adding a set T of triples to A_0 such that each vertex of degree $u - 5$ in L_0 appears in at least one triple in T. This will ensure that in the leave of the resulting system the vertices in U_0 have degree at most $u - 5$ (as required for the leave to be useable). Let $M = \{x \in U_0\colon \deg_{L_0}(x) = u - 5\}$. Clearly, any two distinct elements of M have a common neighbour in L_0 (since $u \geqslant 11$), and so any two elements of M are non-adjacent in L_0 (since (U_0, A_0) is maximal). Hence we have $|M| \leqslant 3$. Choose distinct $m_1, m_2, m_3 \in U_0$ such that $M \subseteq \{m_1, m_2, m_3\}$ and such that there exist two distinct edges $xm_1, ym_2 \in E(L_0)$. Clearly, m_1, m_2 and m_3 exist, since $\epsilon(L_0) \geqslant 4$.

Let $A_0' = A_0 \cup \{\{p, x, m_1\}, \{q, y, m_2\}, \{p, q, m_3\}\}$, and let L_0' denote the leave of the partial extended triple system $(U_0 \cup \{p, q\}, A_0')$. Clearly, $\deg_{L_0'}(v) \leqslant u - 5$ for all $v \in U_0$, and p and q are each incident in L_0' with $u - 5$ edges and one loop. We now define A_1 by adding loops or lollipops to A_0' in such a way that the leave of the partial extended triple system $(U_0 \cup \{p, q\}, A_1)$ is useable and contains no loops. There are three cases to consider, depending on the congruence of $\epsilon(L_0')$.

Case 1. If $\epsilon(L_0') \equiv 0 \pmod 3$, then we let $A_1 = A_0' \cup \{\{p, p, p\}, \{q, q, q\}\}$.

Case 2. If $\epsilon(L_0') \equiv 1 \pmod 3$, then we let $A_1 = A_0' \cup \{\{a, p, p\}, \{q, q, q\}\}$, for some $a \in U_0 \setminus \{m_1, m_3, x\}$.

Case 3. If $\epsilon(L_0') \equiv 2 \pmod 3$, then we let $A_1 = A_0' \cup \{\{a, p, p\}, \{a, q, q\}\}$, for some $a \in U_0 \setminus \{m_1, m_2, m_3, x, y\}$.

Then in all cases, $(U_0 \cup \{p, q\}, A_1)$ is the required partial extended triple system of order u. It is straightforward to verify that its leave L is useable. (Note that since (U_0, A_0) is maximal, Turán's Theorem [15] ensures that $\epsilon(L_0) \leqslant \frac{1}{4}(u - 2)^2$. Thus, $\epsilon(L) \leqslant \epsilon(L_0) + (u - 2) + (u - 1) - 9 < \frac{1}{2}(u(u - 5) - 2)$ since $u \geqslant 11$, as required.) □

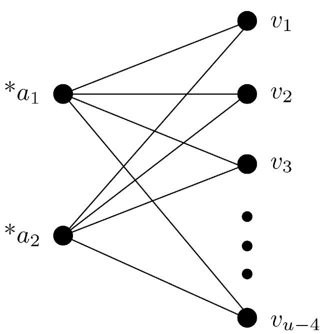
Lemma 6.2. *Let $u \geqslant 11$. Let (U_0, A_0) be a maximal partial extended triple system of order $u - 2$, and let its leave be L_0. If $\beta((U_0, A_0)) = u - 4$, $\epsilon(L_0) \geqslant 3$, $\Delta(L_0) < u - 3$ and $L_0 \neq K_{1,u-4} \cup K_1$, then (U_0, A_0) can be embedded in a partial extended triple system of order u, with a useable leave which contains no loops.*

Proof. Since $\beta((U_0, A_0)) = u - 4$, there are precisely two elements $g_1, g_2 \in U_0$ which are popped in A_0. Let $p, q \notin U_0$. We first consider the partial extended triple system $(U_0 \cup \{p, q\}, A_0)$ and denote its leave by G. Note that for all $x \in U_0$, $\deg_G(x) > u - 5$ if and only if $\deg_{L_0}(x) > u - 7$, and that p and q are each incident in G with $u - 1$ edges and one loop.

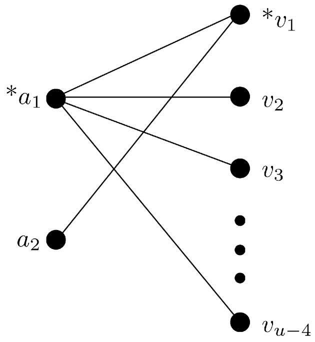
We shall embed (U_0, A_0) in a partial extended triple system $(U_0 \cup \{p, q\}, A_1)$ with a useable leave which contains no loops. To obtain A_1, we shall add triples to A_0 in such a way that the leave L of $(U_0 \cup \{p, q\}, A_1)$ satisfies $\Delta(L) \leqslant u - 5$, $\epsilon(L) \equiv 0 \pmod 3$, $\epsilon(L) \leqslant \frac{1}{2}(u(u - 5) - 2)$ and $\beta((U_0 \cup \{p, q\}, A_1)) = u - 2$. Clearly, p, q and any element $x \in U_0$ satisfying $\deg_{L_0}(x) > u - 7$ will need to appear in (sometimes several) triples of $A_1 \setminus A_0$. There are four cases to consider, depending on $\Delta(L_0)$.

Case 1. $\Delta(L_0) = u - 4$.

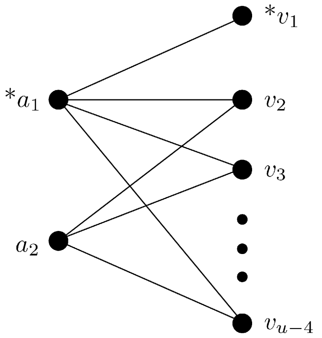
Since (U_0, A_0) is maximal, $\beta((U_0, A_0)) = u - 4$ and $L_0 \neq K_{1,u-4} \cup K_1$, it is straightforward to check that L_0 is one of the three graphs depicted below. The vertices marked with an asterisk are the elements of U_0 which are popped in A_0.



Case 1.1



Case 1.2



Case 1.3

For each graph, there are three cases to consider when defining A_1, depending on the congruence of $\epsilon(L_0)$ (mod 3). In each, we let A_1 consist of A_0 and the triples listed in Table 2.

Case 2. $\Delta(L_0) = u - 5$.

Let $a_1 \in U_0$ satisfy $\deg_{L_0}(a_1) = u - 5$ and let $S = \{x \in U_0\colon \deg_{L_0}(x) \geqslant u - 6\}$. Clearly, any two distinct elements of S have a common neighbour in L_0 (since $u \geqslant 11$), and so any two elements of S are non-adjacent in L_0 (since (U_0, A_0) is maximal). Hence we have $1 \leqslant |S| \leqslant 3$.

If $|S| = 1$, then $S = \{a_1\}$. Clearly there exist distinct edges $a_1v_1, a_1v_2 \in E(L_0)$ such that $v_1, v_2 \notin \{g_1, g_2\}$. Let $A_0' = A_0 \cup \{\{a_1, v_1, p\}, \{a_1, v_2, q\}, \{p, q, v_3\}\}$, for some $v_3 \in U_0 \setminus \{a_1, v_1, v_2, g_1, g_2\}$.

If $|S| = 2$, then $S = \{a_1, a_2\}$ for some $a_2 \in U_0 \setminus \{a_1\}$ with $\deg_{L_0}(a_2) \geqslant u - 6$, and it is clear that there exist edges $a_1v_1, a_2v_2 \in E(L_0)$ such that $v_1 \neq v_2$ and $v_1, v_2 \notin \{g_1, g_2\}$. Let $A_0' = A_0 \cup \{\{a_1, v_1, p\}, \{a_2, v_2, q\}, \{p, q, v_3\}\}$, for some $v_3 \in U_0 \setminus \{a_1, a_2, v_1, v_2, g_1, g_2\}$.

If $|S| = 3$, then $S = \{a_1, a_2, a_3\}$ for some distinct $a_2, a_3 \in U_0 \setminus \{a_1\}$ with $\deg_{L_0}(a_2)$, $\deg_{L_0}(a_3) \geqslant u - 6$, and it is clear that there exist edges $a_1v_1, a_2v_2, a_3v_3 \in E(L_0)$ such that v_1, v_2 and v_3 are distinct and $v_1, v_2, v_3 \notin \{g_1, g_2\}$. Let $A_0' = A_0 \cup \{\{a_1, v_1, p\}, \{a_2, v_2, q\}, \{a_3, v_3, p\}, \{p, q, v_4\}\}$, for some $v_4 \in U_0 \setminus \{a_1, a_2, a_3, v_1, v_2, v_3, g_1, g_2\}$.

Table 2

	$\epsilon(L_0) \equiv 0$ (mod 3)	$\epsilon(L_0) \equiv 1$ (mod 3)	$\epsilon(L_0) \equiv 2$ (mod 3)
Case 1.1	$\{a_1, v_1, p\}, \{a_1, v_2, q\}, \{a_2, v_3, p\}, \{a_2, v_4, q\}, \{p, p, p\}, \{q, q, q\}$	$\{a_1, v_1, p\}, \{a_1, v_2, q\}, \{a_2, v_3, p\}, \{p, q, v_4\}, \{a_2, q, q\}, \{p, p, p\}$	$\{a_1, v_1, p\}, \{a_2, v_2, q\}, \{p, q, v_3\}, \{a_1, q, q\}, \{a_2, p, p\}$
Case 1.2	$\{a_1, v_1, p\}, \{a_1, v_2, q\}, \{p, q, v_3\}, \{p, p, p\}, \{q, q, q\}$	$\{a_1, v_1, p\}, \{a_2, v_1, q\}, \{p, q, v_2\}, \{a_1, q, q\}, \{p, p, p\}$	$\{a_1, v_2, p\}, \{a_2, v_1, q\}, \{p, q, v_3\}, \{a_1, q, q\}, \{v_1, p, p\}$
Case 1.3	$\{a_1, v_1, p\}, \{a_1, v_2, q\}, \{a_2, v_2, p\}, \{p, q, v_3\}, \{p, p, p\}, \{q, q, q\}$	$\{a_1, v_1, p\}, \{a_2, v_2, q\}, \{p, q, v_3\}, \{a_1, q, q\}, \{p, p, p\}$	$\{a_1, v_2, p\}, \{a_2, v_2, q\}, \{p, q, v_3\}, \{a_1, q, q\}, \{v_1, p, p\}$

In all cases, let L_0' denote the leave of the partial extended triple system $(U_0 \cup \{p,q\}, A_0')$. Note that $\deg_{L_0'}(x) \leqslant u-5$ for all $x \in U_0$, p and q are each incident with at most $u-5$ edges and precisely one loop in L_0', and $g_1, g_2 \notin \{a_1, v_i\colon 1 \leqslant i \leqslant |S|\}$. If necessary, we may relabel g_1 and g_2 so that $a_2 \neq g_1$ and $a_3 \neq g_2$, and thus the edges $g_2p, g_1q \in E(L_0')$.

To obtain A_1, we add loops or lollipops to A_0' in such a way that the leave L of the partial extended triple system $(U_0 \cup \{p,q\}, A_1)$ contains no loops and satisfies $\epsilon(L) \equiv 0 \pmod 3$ and $\beta((U_0 \cup \{p,q\}, A_1)) = u-2$. Specifically, if $\epsilon(L_0') \equiv 0 \pmod 3$, $1 \pmod 3$, or $2 \pmod 3$, we let $A_1 = A_0' \cup \{\{p,p,p\},\{q,q,q\}\}$, $A_0' \cup \{\{g_1,p,p\},\{q,q,q\}\}$, or $A_0' \cup \{\{g_1,p,p\},\{g_2,q,q\}\}$, respectively.

Case 3. $\Delta(L_0) = u-6$.

Since $\beta((U_0, A_0)) = u-4$, there are at most two vertices of degree $u-6$ in L_0, and $\{x \in U_0\colon \deg_{L_0}(x) = u-6\} \subseteq \{g_1, g_2\}$. Without loss of generality, suppose $\deg_{L_0}(g_1) = u-6$. Let $a_1 \in U_0 \setminus (\mathrm{Nbd}_{L_0}(g_1) \cup \{g_1\})$ be a vertex satisfying $\deg_{L_0}(a_1) \geqslant \deg_{L_0}(x)$ for all $x \in U_0 \setminus (\mathrm{Nbd}_{L_0}(g_1) \cup \{g_1\})$. Note that $\beta((U_0, A_0)) = u-4$ and parity arguments imply that $L_0 \neq K_{1,u-6} \cup K_3^c$, and so $\deg_{L_0}(a_1) \geqslant 1$. Let $v_1 \in \mathrm{Nbd}_{L_0}(a_1)$, and let $v_2 \in \mathrm{Nbd}_{L_0}(g_1)$, such that $v_2 \notin \{v_1, g_2\}$. Let $v_3 \in U_0 \setminus \{g_1, g_2, v_1, v_2, a_1\}$. Then we let $A_0' = A_0 \cup \{\{a_1,v_1,p\},\{g_1,v_2,q\},\{p,q,v_3\}\}$, and we let L_0' denote the leave of the partial extended triple system $(U_0 \cup \{p,q\}, A_0')$. Note that $\deg_{L_0'}(x) \leqslant u-5$ for all $x \in U_0$ and p and q are each incident with $u-5$ edges and one loop in L_0'.

To obtain A_1, we add loops or lollipops to A_0' in such a way that the leave L of the partial extended triple system $(U_0 \cup \{p,q\}, A_1)$ contains no loops and satisfies $\epsilon(L) \equiv 0 \pmod 3$ and $\beta((U_0 \cup \{p,q\}, A_1)) = u-2$. Specifically, if $\epsilon(L_0') \equiv 0 \pmod 3$, $1 \pmod 3$, or $2 \pmod 3$, we let $A_1 = A_0' \cup \{\{p,p,p\},\{q,q,q\}\}$, $A_0' \cup \{\{g_1,p,p\},\{q,q,q\}\}$, or $A_0' \cup \{\{g_1,p,p\},\{g_2,q,q\}\}$, respectively.

Case 4. $\Delta(L_0) \leqslant u-7$.

Let $a_1a_2, a_3a_4 \in E(L_0)$ such that $a_1a_2, a_3a_4 \neq g_1g_2$, and let $v_1 \in U_0 \setminus \{a_1,a_2,a_3,a_4,g_1,g_2\}$. We let $A_0' = A_0 \cup \{\{a_1,a_2,p\},\{a_3,a_4,q\},\{p,q,v_1\}\}$ and we let L_0' denote the leave of $(U_0 \cup \{p,q\}, A_0')$. Note that $\deg_{L_0'}(x) \leqslant u-5$ for all $x \in U_0$ and p and q are each incident with $u-5$ edges and one loop in L_0'.

To obtain A_1, we add loops or lollipops to A_0' in such a way that the leave L of the partial extended triple system $(U_0 \cup \{p,q\}, A_1)$ contains no loops and also satisfies $\epsilon(L) \equiv 0 \pmod 3$ and $\beta((U_0 \cup \{p,q\}, A_1)) = u-2$. Before doing this, note that if $g_1 \in \{a_1,a_2,a_3,a_4\}$, then without loss of generality $a_1 = g_1$. Furthermore, if $g_2 \in \{a_1,a_2,a_3,a_4\}$, then without loss of generality $a_3 = g_2$, since $a_1a_2, a_3a_4 \neq g_1g_2$, and thus the edges $g_1q, g_2p \in E(L_0')$. If $\epsilon(L_0') \equiv 0 \pmod 3$, $1 \pmod 3$, or $2 \pmod 3$, we let $A_1 = A_0' \cup \{\{p,p,p\},\{q,q,q\}\}$, $A_0' \cup \{\{g_1,q,q\},\{p,p,p\}\}$, or $A_0' \cup \{\{g_1,q,q\},\{g_2,p,p\}\}$, respectively.

It is straightforward to verify in all four cases that the leave L of $(U_0 \cup \{p,q\}, A_1)$ contains no loops and is useable. (Note that since (U_0, A_0) is maximal, Turán's Theorem [15] ensures that $\epsilon(L_0) \leqslant \frac{1}{4}(u-2)^2$. Thus, $\epsilon(L) \leqslant \epsilon(L_0) + (u-2) + (u-1) - 9 < \frac{1}{2}(u(u-5)-2)$ since $u \geqslant 11$, as required.) $\square$

Lemma 6.3. *Let $u \geqslant 11$. Let (U_0, A_0) be a maximal partial extended triple system of order $u-2$, and let its leave be L_0. If $\beta((U_0, A_0)) \leqslant u-6$, L_0 is non-empty and $\Delta(L_0) < u-3$, then*

(U_0, A_0) can be embedded in a partial extended triple system $(U_0 \cup \{p, s\}, A_1)$ of order u with a leave L_1, such that L_1 contains no loops and is the union of two edge-disjoint graphs L and S where L is useable and every edge of S is incident with s.

Proof. The construction is quite simple, but tedious. Let $p, s \notin U_0$. We first embed (U_0, A_0) in a system $(U_0 \cup \{p\}, A_0')$, where A_0' is obtained from A_0 by adding certain normal triples.

Let $R = \{x \in U_0\colon \deg_{L_0}(x) \geqslant u-5\}$, and let $r = |R|$. If $r = 0$, then let $a_1v_1 \in E(L_0)$ and let $A_0' = A_0 \cup \{\{a_1, v_1, p\}\}$. Otherwise, denote the elements of R by $a_1, a_2, \dots, a_r$. Clearly, any two distinct elements a_i, a_j have a common neighbour in L_0 (since $u \geqslant 11$), and so a_i and a_j are not adjacent in L_0 (since (U_0, A_0) is maximal). Hence we have $r \leqslant 3$. Thus, (since $\deg_{L_0}(a_i) \geqslant 6$ for $i = 1, 2, \dots, r$) there are distinct vertices $v_1, v_2, \dots, v_r \in U_0$ such that $a_1v_1, a_2v_2, \dots, a_rv_r \in E(L_0)$, and we let $A_0' = A_0 \cup \{\{a_i, v_i, p\}\colon i = 1, 2, \dots, r\}$.

Let L_0' denote the leave of $(U_0 \cup \{p\}, A_0')$. We note the following properties of $(U_0 \cup \{p\}, A_0')$.

- For each x in U_0, x is unpopped (popped) in (U_0, A_0) if and only if x is unpopped (popped) in $(U_0 \cup \{p\}, A_0')$.
- $\deg_{L_0'}(x) \leqslant u-5$ for all $x \in U_0$ (by construction, since $\Delta(L_0) < u-3$).
- $\deg_{L_0'}(x) \equiv \deg_{L_0}(x) + 1 \pmod 2$ for all $x \in U_0$, and so all elements of U_0 which are unpopped in A_0' have degree in L_0' at most $u-6$.
- $\deg_{L_0'}(p) \leqslant u-4$.

We next embed $(U_0 \cup \{p\}, A_0')$ in a partial extended triple system $(U_0 \cup \{p\}, A_0'')$, where A_0'' is obtained from A_0' by adding a single lollipop $\{b, p, p\}$ where $b \in U_0$. The element b will be chosen so that $1 \leqslant \beta((U_0 \cup \{p\}, A_0'')) \leqslant u-7$. (The significance of this property will become clear later.)

There is one special case to consider when choosing b, and we deal with this separately first. Suppose $\beta((U_0, A_0)) = 1$. Then we choose

$$b \in \bigl\{x \in U_0 \setminus \{a_i, v_i\colon i = 1, 2, \dots, r\}\colon x \text{ is popped in } (U_0, A_0)\bigr\},$$

and we let $A_0'' = A_0' \cup \{b, p, p\}$. Clearly, $\beta((U_0 \cup \{p\}, A_0'')) = 2$ (the two unpopped vertices being b and the unique vertex in U_0 which is unpopped in A_0), and so $1 \leqslant \beta((U_0 \cup \{p\}, A_0'')) \leqslant u-7$ as required.

We now move to the case $\beta((U_0, A_0)) \neq 1$. We let

$$Q = \bigl\{x \in U_0 \setminus \{a_i, v_i\colon i = 1, 2, \dots, r\}\colon x \text{ is unpopped in } (U_0, A_0)\bigr\}.$$

If $Q \neq \emptyset$ then we choose $b \in Q$. Otherwise, we let $b \in U_0 \setminus \{a_i, v_i\colon i = 1, 2, \dots, r\}$ (so b is popped in (U_0, A_0)). Let $A_0'' = A_0' \cup \{\{b, p, p\}\}$. We need to check that $1 \leqslant \beta((U_0 \cup \{p\}, A_0'')) \leqslant u-7$.

If $Q \neq \emptyset$, then $\beta((U_0 \cup \{p\}, A_0'')) = \beta((U_0, A_0)) - 1 \leqslant u-7$. Furthermore, the case $\beta((U_0, A_0)) = 1$ was dealt with above, and so $1 \leqslant \beta((U_0 \cup \{p\}, A_0''))$ as required.

Hence we can assume that $Q = \emptyset$. In this case $\beta((U_0 \cup \{p\}, A_0'')) = \beta((U_0, A_0)) + 1 \geqslant 1$, so we need only check that $\beta((U_0 \cup \{p\}, A_0'')) \leqslant u-7$. If $\beta((U_0, A_0)) \leqslant u-8$ the claim holds, so we can assume that $\beta((U_0, A_0)) = u-6$ (recall that the number of unpopped elements has the same parity as the order). Since $Q = \emptyset$, all elements of U_0 which are unpopped in A_0 occur in $\{a_1, a_2, \dots, a_r, v_1, v_2, \dots, v_r\}$, and since $\beta((U_0, A_0)) = u-6$, there are precisely four elements of U_0 which are popped in A_0. Hence we have $u-2 \leqslant 2r+4$. Since $u \geqslant 11$, we have $r \geqslant 3$. Thus $r = 3$ (recall that $r \leqslant 3$), and so there are three vertices (a_1, a_2 and a_3) in U_0 with degree in L_0 at least $u-5$. We have already noted that these are pairwise non-adjacent in L_0 and thus $L_0 = K_{3,u-5}$. But this contradicts $\beta((U_0, A_0)) = u-6$, and so the claim holds.

Thus we have established that the choices of b defined above do indeed ensure that $1 \leqslant \beta((U_0 \cup \{p\}, A_0'')) \leqslant u-7$. Let L_0'' denote the leave of $(U_0 \cup \{p\}, A_0'')$. We note several other properties of $(U_0 \cup \{p\}, A_0'')$.

- $\deg_{L_0''}(x) \leqslant u-5$ for all $x \in U_0 \cup \{p\}$.
- All elements of $U_0 \cup \{p\}$ which are unpopped (popped) in A_0'' have degree in L_0'' congruent to $u \pmod 2$ ($u-1 \pmod 2$).
- All elements of $U_0 \cup \{p\}$ which are unpopped in A_0'' have degree in L_0'' at most $u-6$.
- $\beta((U_0 \cup \{p\}, A_0'')) \equiv u-1 \pmod 2$.

Finally, we embed $(U_0 \cup \{p\}, A_0'')$ in the partial extended triple system $(U_0 \cup \{p, s\}, A_1)$, where $A_1 = A_0'' \cup \{\{s, s, s\}\}$, to obtain the required embedding of (U_0, A_0). We denote the leave of $(U_0 \cup \{p, s\}, A_1)$ by L_1.

In order to define the edge-disjoint graphs L and S, we first define a graph H with $V(H) = U_0 \cup \{p, s\}$, and $E(H) = E(L_0'') \cup \{sx\colon\ x \text{ is unpopped in } (U_0 \cup \{p\}, A_0'')\}$. The following properties of H follow immediately from the properties of $(U_0 \cup \{p\}, A_0'')$.

- $1 \leqslant \deg_H(s) \leqslant u-7$ and $p \notin \mathrm{Nbd}_H(s)$.
- $\Delta(H) \leqslant u-5$.
- For all $x \in U_0 \cup \{p, s\}$, $\deg_H(x) \equiv u-1 \pmod 2$.

We are now ready to define the edge-disjoint graphs L and S. Let $V(L) = V(S) = U_0 \cup \{p, s\}$. There are three cases to consider, depending on the congruence of $\epsilon(H) \pmod 3$. In all three we define $E(L)$ and then define $E(S)$ by $E(S) = E(L_1) \setminus E(L)$.

Case 1. If $\epsilon(H) \equiv 0 \pmod 3$, then we let $E(L) = E(H)$.

Case 2. If $\epsilon(H) \equiv 1 \pmod 3$, then we let $E(L) = E(H) \setminus \{sc\}$, where $c \in U_0$ is any vertex adjacent in H to s.

Case 3. If $\epsilon(H) \equiv 2 \pmod 3$, then we let $E(L) = E(H) \cup \{sc\}$, where $c \in U_0$ is a vertex not adjacent to s in H, with $\deg_{L_0''}(c) \leqslant u-7$ (so that $\deg_L(c) \leqslant u-5$).

For Case 3 we need to check that c exists. Since $\deg_H(s) \leqslant u-7$, there are at least five vertices in U_0 not adjacent in H to s. By definition of H, these are popped in A_0''. Suppose that each of these vertices has degree in L_0'' at least $u-5$. Then they each have degree in L_0 at least $u-6$. However, there are at most four vertices in U_0 with degree in L_0 at least $u-6$ (since (U_0, A_0) is maximal). It follows that there is at least one vertex $c \in U_0$ not adjacent to s in H, with $\deg_{L_0''}(c) \leqslant u-7$.

In all three cases it is clear that every edge of S is incident with s, and it is also straightforward to check that L is useable. (Note that since (U_0, A_0) is maximal, Turán's Theorem [15] ensures that $\epsilon(L_0) \leqslant \frac{1}{4}(u-2)^2$. Thus, $\epsilon(L) \leqslant \epsilon(L_0) + (u-2) - 3 - 1 + (u-7) + 1 < \frac{1}{2}(u(u-5)-2)$ since $u \geqslant 11$, as required.) $\square$

Remark. From the construction in the preceding proof, it is clear that if there are two elements of $U_0 \cup \{p, s\}$ with degree in L congruent to u (mod 2), then these are s and the vertex c defined in Cases 2 and 3.

7. Repacking partial Steiner triple systems

As mentioned earlier, our method relies on the techniques for repacking partial Steiner triple systems that were introduced in [3]. In this section we define the relevant concepts, restate some repacking results from [3,5], and prove some additional repacking results. These results will be used in Sections 8 and 9.

In [3], the following definition of a *repacking* of the triples of a partial Steiner triple system was introduced.

Definition 7.1. [3] Let U and W be disjoint sets, let $(U \cup W, B)$ be a partial Steiner triple system, and let G be its leave. A *repacking* of $(U \cup W, B)$ is a partial Steiner triple system $(U \cup W, B')$ with leave G' satisfying the following conditions.

- $|B'| = |B|$.
- $G'(U) = G(U)$.
- For all $x, y, z \in U$, $\{x, y, z\} \in B'$ if and only if $\{x, y, z\} \in B$.
- For all $x \in U$, $\deg_{G'}(x) = \deg_G(x)$.

The notion of repacking depends on the partition of the underlying set of the partial Steiner triple system, and this will always be made clear. The underlying set is always given as the union of two sets and these define the partition. Clearly, the basic property of repacking is that if (U, A) is embedded in $(U \cup W, B)$ and $(U \cup W, B')$ is a repacking of $(U \cup W, B)$, then (U, A) is embedded in $(U \cup W, B')$. Moreover, it is easy to see that if $(U \cup W, B')$ with leave G' is a repacking of $(U \cup W, B)$ with leave G then the following conditions also hold.

- $\epsilon(G'(U)) = \epsilon(G(U))$, $\epsilon(G'(U, W)) = \epsilon(G(U, W))$, and $\epsilon(G'(W)) = \epsilon(G(W))$.
- For $i = 0, 1, 2, 3$, the number of U-type i triples in B' is the same as the number of U-type i triples in B. Furthermore, for each $x \in U$ the number of U-type i triples in B' that contain x is the same as the number of U-type i triples in B that contain x.

Let (V, B) be a partial Steiner triple system, let H be its underlying graph, let $a, b \in V$, and let $P = x_1, x_2, \dots, x_t$ be a maximal alternating (a, b)-path in H, where P is not an alternating cycle of H with an edge removed. The partial Steiner triple system (V, B') obtained from (V, B) by interchanging a with b in the triples containing the pairs $\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{t-1}, x_t\}$ is called the partial Steiner triple system obtained from (V, B) by *switching along the alternating (a, b)-path from x_1 to x_t*.

Let U and W be disjoint sets, let $(U \cup W, B)$ be a partial Steiner triple system, and let G be its leave. If $a, b \in W$ then it is easy to see that the partial Steiner triple system $(U \cup W, B')$ obtained from $(U \cup W, B)$ by switching along an (a, b)-path is a repacking of $(U \cup W, B)$. This repacking technique was introduced in [1] and used to settle the existence problem for *equitable* partial Steiner triple systems. The following result from [5] is also easily proved using this method.

Lemma 7.2. [5] *Let U and W be disjoint sets, let $(U \cup W, B)$ be a partial Steiner triple system, and let G be its leave. Then there exists a repacking of $(U \cup W, B)$ with a leave G' such that for all $x, y \in W$, $|\deg_{G'}(x) - \deg_{G'}(y)| \leqslant 2$.*

In [3], the following two techniques for repacking triples in partial Steiner triple systems were introduced.

Lemma 7.3. [3] *Let U and W be disjoint sets, let $(U \cup W, B)$ be a partial Steiner triple system, and let G be its leave. Suppose G contains a connected subgraph M with $V(M) \subseteq W$ and where M is neither a cycle nor a tree. Then there exists a partial Steiner triple system $(U \cup W, B')$ with a U-type 0 triple $T \in B'$ and such that $(U \cup W, B' \setminus \{T\})$ is a repacking of $(U \cup W, B)$. Moreover, $T \subseteq V(M)$ and any edge used in exactly one of $(U \cup W, B)$ and $(U \cup W, B' \setminus \{T\})$ is incident with a vertex in $V(M)$.*

Lemma 7.4. [3] *Let U and W be disjoint sets, let $(U \cup W, B)$ be a partial Steiner triple system, let G be its leave, and suppose the following conditions hold.*

(1) *There exists a path a_1, b_1, c in G with $a_1 \in W$ and $b_1 \in U$.*
(2) *If $\{c, x, y\} \in B$ then at least one of x and y is in U.*
(3) *For all $x \in U$, $\deg_{G(U,W)}(x) \geqslant 1$.*
(4) *There exists an edge $ca_0 \in E(G)$ with $a_0 \in W$.*

Then there exists a partial Steiner triple system $(U \cup W, B')$ with $\{a_1, b_1, c\} \in B'$ and such that $(U \cup W, B' \setminus \{\{a_1, b_1, c\}\})$ is a repacking of $(U \cup W, B)$.

Remark. Lemma 7.4 applies both when $c \in U$ and when $c \in W$.

The following three repacking lemmas will be used in Section 8.

Lemma 7.5. *Let U and W be disjoint sets with $|U| = u$ and $|W| = w$, let $(U \cup W, B)$ be a partial Steiner triple system of order $u + w$, let G be its leave and suppose $G(W)$ is 2-regular. Then there exists a partial Steiner triple system $(U \cup W, B')$ with a leave G' such that either*

(1) *$(U \cup W, B')$ is a repacking of $(U \cup W, B)$ and $G'(W)$ is a w-cycle; or*
(2) *B' contains a U-type 0 triple T such that $(U \cup W, B' \setminus \{T\})$ is a repacking of $(U \cup W, B)$.*

Proof. If $G(W)$ is connected then we are finished, so assume that $G(W)$ has two or more components. Let $a, b \in W$ be vertices in different components of $G(W)$. Let c be a neighbour of a in $G(W)$ and let d_1 and d_2 be the two neighbours of b in $G(W)$. Consider the alternating (a, b)-path in the underlying graph of $(U \cup W, B)$ that emanates from c. Let t be the vertex at which the path terminates (note that $t \neq c$). Let $(U \cup W, B_1)$ be the repacking of $(U \cup W, B)$ obtained by switching along the alternating (a, b)-path from c to t, and let G_1 be its leave. Note that $cb \in E(G_1)$ but that $ca \notin E(G_1)$.

If $t \notin \{d_1, d_2\}$, then b is in a component of G_1 which is neither a cycle nor a tree. Thus we may apply Lemma 7.3 to obtain a partial Steiner triple system $(U \cup W, B')$ with a U-type 0 triple $T \in B'$ such that $(U \cup W, B' \setminus \{T\})$ is a repacking of $(U \cup W, B)$. In this case, Condition (2) of the lemma is satisfied.

Hence we can assume that $t \in \{d_1, d_2\}$. Then without loss of generality $t = d_1$ and G_1 is obtained from G by replacing the edges ca and d_1b with the edges cb and d_1a. Thus the vertices in the component of $G(W)$ containing a and the vertices in the component of $G(W)$ containing b form the vertex set of a single component of $G_1(W)$, and each other component of $G(W)$ is also a component of $G_1(W)$. So $G_1(W)$ is 2-regular and has fewer components than $G(W)$. Since there are only finitely many components of $G(W)$, we can repeat this process until we obtain a partial Steiner triple system satisfying Condition (1) or Condition (2). □

Lemma 7.6. *Let U and W be disjoint sets with $|U| = u$ and $|W| = w$, let $(U \cup W, B)$ be a partial Steiner triple system of order $u + w$, let G be its leave and suppose $G(W)$ is a w-cycle. If B contains at least one U-type 0 triple T, then there exists a partial Steiner triple system $(U \cup W, B')$ with two U-type 0 triples T_1 and T_2 such that $(U \cup W, B' \setminus \{T_1, T_2\})$ is a repacking of $(U \cup W, B \setminus \{T\})$.*

Proof. Let $T = \{a, b, c\}$ and let $G(W)$ be the cycle $(v_1, v_2, \ldots, v_w)$. Then without loss of generality, $a = v_1$, $b = v_i$ for some $i \in \{3, 4, \ldots, w-3\}$ and $c = v_j$ for some $j \in \{i+2, i+3, \ldots, w-1\}$. Denote the leave of the partial Steiner triple system $(U \cup W, B \setminus \{\{a, b, c\}\})$ by G_0. Then G_0 contains the subgraph M consisting of the cycle $(v_1, v_2, \ldots, v_i)$ and the pendant edge $v_i v_{i+1}$. We apply Lemma 7.3 and denote the leave of the resulting partial Steiner triple system by G'. Then $\deg_{G'(W)}(c) = 4$ and $\deg_{G'(W)}(x) \equiv 0 \pmod 2$ for every $x \in W$. Thus, it is clear that $G'(W)$ contains a subgraph M' which is neither a cycle nor a tree, and so we may apply Lemma 7.3 to obtain the required partial Steiner triple system. □

Lemma 7.7. *Let U and W be disjoint sets with $|U| = u$ and $|W| = w$ and let $(U \cup W, B)$ be a partial Steiner triple system of order $u + w$. Let G be its leave and suppose that $G(W)$ is a w-cycle and $\deg_{G(U,W)}(x) \geqslant 1$ for all $x \in U$. If B contains at least one U-type 1 triple and does not contain any U-type 0 triples, then there exists a partial Steiner triple system $(U \cup W, B')$ with a U-type 0 triple T such that $(U \cup W, B' \setminus \{T\})$ is a repacking of $(U \cup W, B)$.*

Proof. Let $\{a, b, c\} \in B$, where $a \in U$ and $b, c \in W$. Since $\deg_{G(U,W)}(x) \geqslant 1$ for all $x \in U$, there is a vertex $d \in W$ adjacent to a in G. Consider the system $(U \cup W, B \setminus \{\{a, b, c\}\})$ and let its leave be G_0.

In the special case where (b, d, c) is a cycle in $G_0(W)$, let $B_1 = (B \setminus \{\{a, b, c\}\}) \cup \{\{a, b, d\}\}$ and let the leave of $(U \cup W, B_1)$ be G_1. Then $G_1(W)$ contains a subgraph which is neither a cycle nor a tree. Applying Lemma 7.3 to $(U \cup W, B_1)$ yields the required partial Steiner triple system.

Hence we may assume that (b, d, c) is not a cycle in $G_0(W)$. Then it is clear that $G_0(W)$ contains a subgraph M, with $d \notin V(M)$, which is neither a cycle nor a tree. We apply Lemma 7.3 to obtain a partial Steiner triple system $(U \cup W, B_1)$ with a U-type 0 triple T, such that $(U \cup W, B_1 \setminus \{T\})$ is a repacking of $(U \cup W, B \setminus \{\{a, b, c\}\})$. We denote the leave of $(U \cup W, B_1)$ by G_1. Note that $d \notin T$ (see Lemma 7.3), and thus if $\{d, x, y\} \in B_1$ then at least one of x and y is in U. Furthermore, $\deg_{G_1(W)}(d) = 2$ and G_1 contains the path d, a, f for some $f \in W$. Thus, we may apply Lemma 7.4 with $a_1 = f$, $b_1 = a$, c and $a_0 \in \mathrm{Nbd}_{G_1(W)}(d)$ to obtain the required partial Steiner triple system. □

8. Partial Steiner triple systems with big leaves

In this section we show that any partial Steiner triple system (U, A) of order u with a leave L_1 containing a spanning useable subgraph L with $\epsilon(L) \geqslant 3u - 18$ has an embedding in a partial Steiner triple system of order $2u + 1$ with a leave H such that either

- $E(H) = E(L_1) \setminus E(L)$; or
- $E(H)$ consists of $E(L_1) \setminus E(L)$ and the edges of a path of length three, with endvertices in U and internal vertices not in U.

There are several steps to the embedding. We begin with the following easy corollary of Lemma 7.4, proven in [3], which is needed for the proof of Lemma 8.2.

Lemma 8.1. [3] *Let U and W be disjoint sets, let $(U \cup W, B)$ be a partial Steiner triple system, let G be its leave, and suppose the following conditions hold.*

(1) *There exists a path a_1, b_1, c in G with $a_1, c \in W$ and $b_1 \in U$.*
(2) *There are no U-type 0 triples in B.*
(3) *For all $x \in U$, $\deg_{G(U,W)}(x) \geqslant 1$.*
(4) *There exists an edge $ca_0 \in E(G)$ with $a_0 \in W$, or there exists a vertex $d \in W$ with $\deg_{G(W)}(d) \geqslant 2$.*

Then there exists a partial Steiner triple system $(U \cup W, B')$ with $\{a_1, b_1, c\} \in B'$ and such that $(U \cup W, B' \setminus \{\{a_1, b_1, c\}\})$ is a repacking of $(U \cup W, B)$.

We are now ready to carry out the first part of the embedding for this section. To do this we will begin by adding a set W of $u - 4$ new elements to the given partial Steiner triple system (U, A), and then use Vizing's Theorem [16] to add a set of U-type 2 triples which use the edges of a spanning useable subgraph of the leave of (U, A). We will then use Lemma 8.1 to add U-type 1 triples until the number of unused edges with one vertex in U and one vertex in W is $u + 2$.

Lemma 8.2. *Let (U, A) be a partial Steiner triple system of order $u \geqslant 11$, let L_1 be its leave and suppose that L_1 contains a spanning useable subgraph L with $\epsilon(L) \geqslant 3u - 18$. Then (U, A) can be embedded in a partial Steiner triple system $(U \cup W, B)$ of order $u + w$ with a leave G where*

- *$w = |W| = u - 4$,*
- *$G(U) = L_1 - L$,*
- *$\epsilon(G(U, W)) = u + 2$,*
- *$\epsilon(G(W)) = \epsilon(L) - 2u + 11$, and*
- *$\deg_{G(U,W)}(x) \geqslant 1$ for all $x \in U$.*

Proof. Let $w = u - 4$ and let W be a set with $|W| = w$ and $W \cap U = \emptyset$. Since L is useable, we have $\Delta(L) \leqslant w - 1$, and thus Vizing's Theorem [16] guarantees that there exists a proper edge colouring γ of L with colour set W. Let

$$B_0 = A \cup \big\{\{x, y, \gamma(xy)\}: xy \in E(L)\big\}$$

so that $(U \cup W, B_0)$ is a partial Steiner triple system. Let G_0 be its leave. We now construct a sequence $(U \cup W, B_0), (U \cup W, B_1), \ldots, (U \cup W, B_t)$ of partial Steiner triple systems, with leaves $G_0, G_1, \ldots, G_t$ say, where $t = \frac{1}{2}(u(u-4) - 2\epsilon(L) - u - 2)$ and where for $i = 0, 1, \ldots, t-1$, $(U \cup W, B_{i+1})$ is obtained from $(U \cup W, B_i)$ by applying Lemma 8.1 with b_1 chosen such that $\deg_{G_i(U,W)}(b_1) \geqslant 3$. Since L is useable, $\epsilon(L) \leqslant \frac{1}{2}(u(u-5) - 2)$ and so $t \geqslant 0$.

To check that we can indeed construct this sequence of partial Steiner triple systems, we let $k \in \{0, 1, \ldots, t-1\}$, assume we have constructed $(U \cup W, B_k)$, and show that it satisfies the conditions of Lemma 8.1 (and moreover, that there is a suitable b_1 with $\deg_{G_k(U,W)}(b_1) \geqslant 3$). Note that

- the U-type 3 triples in B_k are precisely those in A;
- there are exactly $\epsilon(L)$ U-type 2 triples in B_k;
- there are exactly k U-type 1 triples in B_k; and
- there are no U-type 0 triples in B_k.

Also note that $G_k(U) = L_1 - L$, $\epsilon(G_k(U, W)) = u(u-4) - 2\epsilon(L) - 2k = u + 2(t - k + 1)$, and $\epsilon(G_k(W)) = \binom{w}{2} - k = \epsilon(L) - 2u + 11 + (t - k)$. Since $k \leqslant t - 1$, we have $\epsilon(G_k(U, W)) \geqslant u + 4$ and $\epsilon(G_k(W)) \geqslant \epsilon(L) - 2u + 12$.

We now show that $(U \cup W, B_k)$ satisfies Condition (1) of Lemma 8.1, and that there is a suitable b_1 with $\deg_{G_k(U,W)}(b_1) \geqslant 3$. Since L is useable, there are at most two vertices in U with degree in L congruent to u (mod 2), and hence there are at most two vertices in U with even degree in $G_k(U, W)$. Since $\epsilon(G_k(U, W)) \geqslant u + 4$, G_k has a path a_1, b_1, c with $a_1, c \in W$ and $b_1 \in U$, and moreover the vertex $b_1 \in U$ may be chosen so that $\deg_{G_k(U,W)}(b_1) \geqslant 3$, as required.

Condition (2) of Lemma 8.1 is clearly satisfied. There are two cases to consider to show that Condition (3) of Lemma 8.1 holds. Firstly, if $x \in U$ satisfies $\deg_L(x) \equiv u - 1 \pmod 2$, then $\deg_{G_k(U,W)}(x) \equiv 1 \pmod 2$, and so $\deg_{G_k(U,W)}(x) \geqslant 1$. Secondly, if $x \in U$ satisfies $\deg_L(x) \equiv u \pmod 2$, then $\deg_L(x) \leqslant u - 6$ and so $\deg_{G_0(U,W)}(x) \geqslant 2$. This implies that $\deg_{G_k(U,W)}(x) \geqslant 2$ for all $k \in \{1, 2, \ldots, t-1\}$, since B_k is obtained from B_{k-1} by applying Lemma 8.1 with $\deg_{G_{k-1}(U,W)}(b_1) \geqslant 3$. So Condition (3) of Lemma 8.1 does indeed hold.

If a_1 and c are not both isolated vertices in $G_k(W)$ then, by interchanging the labels of a_1 and c if necessary, Condition (4) of Lemma 8.1 is satisfied. We now show that Condition (4) holds when a_1 and c are both isolated vertices in $G_k(W)$. Since $k \leqslant t - 1$, $G_k(W)$ has at least $\epsilon(L) - 2u + 12$ edges. But $\epsilon(L) \geqslant 3u - 18$ and so $G_k(W)$ has at least $u - 6 = w - 2$ edges. It follows that there exists a vertex $d \in W$ with $\deg_{G_k(W)}(d) \geqslant 2$ and Condition (4) of Lemma 8.1 is satisfied.

Hence we can construct $(U \cup W, B_t)$. If we let $B = B_t$ then it is clear that $(U \cup W, B)$ is the required partial Steiner triple system. □

The following lemma uses Lemmas 7.3, 7.5–7.7 to add U-type 0 triples to the embedding of (U_0, A_0) obtained in Lemma 8.2.

Lemma 8.3. *Let (U, A) be a partial Steiner triple system of order $u \geqslant 11$, let L_1 be its leave and suppose that L_1 contains a spanning useable subgraph L with $\epsilon(L) \geqslant 3u - 18$. Then (U, A) can be embedded in a partial Steiner triple system $(U \cup W, B)$ of order $u + w$ with a leave H satisfying the following conditions.*

(1) $w = |W| = u - 4$.
(2) $H(U) = L_1 - L$.

(3) $\deg_{H(U,W)}(x) \geqslant 1$ *for every* $x \in U$.
(4) $\epsilon(H(U, W)) = u + 2$.
(5) $\epsilon(H(W)) = u - 7$.

Proof. By Lemma 8.2, we can embed (U, A) in a partial Steiner triple system $(U \cup W, B_1)$ of order $u + w$ with a leave G satisfying Conditions (1)–(4) and satisfying $\epsilon(G(W)) = \epsilon(L) - 2u + 11$.

In the special case where $\epsilon(L) = 3u - 18$, we have $\epsilon(G(W)) = u - 7$ and we are finished. For $\epsilon(L) \geqslant 3u - 15$ we have $\epsilon(G(W)) \geqslant u - 4$ and we will proceed by using Lemma 7.3 to add U-type 0 triples until the number of unused edges with both vertices in W is $u - 4$. We will then apply Lemmas 7.3, 7.5–7.7 as appropriate, to repack (possibly trivially) and add a further U-type 0 triple to the system, so that the number of unused edges with both vertices in W is $u - 7$ as required.

If $\epsilon(G(W)) = u - 4$ then let $B^* = B_1$ and let $G^* = G$. If $\epsilon(G(W)) > u - 4$ then $\epsilon(G(W)) = u - 4 + 3t$ for some positive integer t. Thus, $\epsilon(G(W)) \geqslant u - 1 = w + 3$ and so $G(W)$ contains a connected subgraph which is neither a cycle nor a tree. Hence it is clear that we can repeatedly apply Lemma 7.3 until we obtain a partial Steiner triple system $(U \cup W, B^*)$ with a leave G^* such that there are exactly $w = u - 4$ edges in $G^*(W)$ (and $u + 2$ edges in $G^*(U, W)$ and such that $G^*(U) = L_1 - L$).

Since $G^*(W)$ has w vertices and w edges, it is clear that either $G^*(W)$ contains a component which is neither a cycle nor a tree, or $G^*(W)$ is 2-regular. In the first case we apply Lemma 7.3 to obtain a partial Steiner triple system with the required properties. In the second case, we apply Lemma 7.5 to obtain either the required partial Steiner triple system, or a partial Steiner triple system $(U \cup W, B^\dagger)$ with a leave $G^\dagger$ such that $G^\dagger(W)$ is a w-cycle. In the latter case, we obtain the required partial Steiner triple system using either Lemma 7.6 or Lemma 7.7. To see that we can indeed do this, observe that if $B^\dagger$ contains a U-type 0 triple then we can apply Lemma 7.6. Otherwise we can apply Lemma 7.7 (in this case it is clear that $B^\dagger$ contains at least one U-type 1 triple). $\square$

Before proving the main result of this section, we need the result given below in Lemma 8.6, concerning repackings and proper edge colourings of certain subgraphs of leaves. To prove it, we use the following two lemmas from [3]. The second is actually slightly different to the lemma from [3], but the proof is essentially the same.

Lemma 8.4. [3] *Let G be a graph with $\Delta(G) \leqslant 3$ and $\chi'(G) > 3$. Then G has at least two vertices of degree 3, each of which is incident with no bridge.*

Lemma 8.5. [3] *Let u be an integer and let $w = u - 4$. Let U and W be disjoint sets with $|U| = u$ and $|W| = w$, let $(U \cup W, B)$ be a partial Steiner triple system, let G be its leave, and suppose the following conditions are satisfied.*

(1) $\epsilon(G(U, W)) = u + 2$.
(2) $\epsilon(G(W)) = u - 7$.

Then there exists a repacking $(U \cup W, B')$ of $(U \cup W, B)$ with a leave G' such that $\deg_{G'}(x) = 3$ *for all* $x \in W$, *and* $\chi'(G' - G'(U)) = 3$.

Lemma 8.6. *Let u be an integer and let $w = u - 4$. Let U and W be disjoint sets with $|U| = u$ and $|W| = w$, let $(U \cup W, B)$ be a partial Steiner triple system, let G be its leave, and suppose the following conditions are satisfied.*

(1) $\deg_{G(U,W)}(x) \geqslant 1$ *for every* $x \in U$.
(2) $\epsilon(G(U, W)) = u + 2$.
(3) $\epsilon(G(W)) = u - 7$.

Then there exists a repacking $(U \cup W, B')$ of $(U \cup W, B)$ with a leave G' such that $\deg_{G'}(x) = 3$ for all $x \in W$, and $\chi'(G' - G'(U)) = 3$.

Proof. First note that Conditions (1) and (2) imply that either

- there is one vertex $\alpha \in U$ with $\deg_{G(U,W)}(\alpha) = 3$ and $\deg_{G(U,W)}(x) = 1$ for every $x \in U \setminus \{\alpha\}$; or
- there are two vertices $\beta, \gamma \in U$ with
$$\deg_{G(U,W)}(\beta) = \deg_{G(U,W)}(\gamma) = 2 \quad \text{and} \quad \deg_{G(U,W)}(x) = 1$$
for every $x \in U \setminus \{\beta, \gamma\}$.

In the first case, the graph $G - G(U)$ satisfies the conditions of Lemma 8.5, and we are finished. Hence, we need only consider the second case.

Let $(U \cup W, B_0)$ be the repacking of $(U \cup W, B)$ obtained by applying Lemma 7.2 and let G_0 be its leave. Since $w = u - 4$, $G(U, W)$ has $u + 2$ edges, and $G(W)$ has $u - 7$ edges, it follows that every vertex of W has degree 3 in G_0. Clearly, it is sufficient to show that there exists a repacking $(U \cup W, B')$ of $(U \cup W, B_0)$ with a leave G' such that $\chi'(G'(W) \cup G'(\{\beta, \gamma\}, W)) \leqslant 3$ (any proper edge colouring of $G'(W) \cup G'(\{\beta, \gamma\}, W)$ with three colours can be extended to a proper edge colouring of $G' - G'(U)$ with three colours, as $\Delta(G' - G'(U)) = 3$ and all the vertices in $U \setminus \{\beta, \gamma\}$ have degree 1 in $G' - G'(U)$).

Let $H_0 = G_0(W) \cup G_0(\{\beta, \gamma\}, W)$. So H_0 has $w + 2$ vertices, $w + 1$ edges, and $\Delta(H_0) = 3$. Note that any such graph has a (possibly trivial) component which is a tree. Let T be a tree in H_0. If $\chi'(H_0) = 3$ we are finished so assume $\chi'(H_0) > 3$. Then by Lemma 8.4, H_0 has at least two vertices of degree 3, each of which is incident with no bridge. Let a be one of these vertices and let $b \notin \{\beta, \gamma\}$ be a vertex in $V(T)$. Note that the component containing a is not a tree so $b \neq a$, and $a \notin \{\beta, \gamma\}$, since $\deg_{H_0}(\beta) = \deg_{H_0}(\gamma) = 2$.

Consider the alternating (a, b)-paths in the underlying graph of $(U \cup W, B_0)$ that emanate from the three neighbours of a in H_0. At least one of these paths must terminate at a neighbour of b in G_0. Let a' and b' be the endvertices of such a path such that $aa' \in E(H_0)$ and $bb' \in E(G_0)$. Let $(U \cup W, B_1)$ be the repacking of $(U \cup W, B_0)$ obtained by switching along the alternating (a, b)-path from a' to b', let G_1 be its leave, and let $H_1 = G_1(W) \cup G_1(\{\beta, \gamma\}, W)$. So G_1 is obtained from G_0 by replacing the edges aa' and bb' with ab' and ba'. Since aa' is not a bridge, the vertices in the component of H_0 containing a and the vertices in the component of H_0 containing b form the vertex set of a single component of H_1, and any other components of H_0 are also components of H_1. So H_1 has fewer components than H_0. Since there are only finitely many components of H_0, we can repeat this process until we obtain the required repacking. □

We are now ready to prove the main result of this section.

Lemma 8.7. *Let (U, A) be a partial Steiner triple system of order $u \geqslant 11$, let L_1 be its leave and suppose that L_1 contains a spanning useable subgraph L with $\epsilon(L) \geqslant 3u - 18$. Then (U, A) can be embedded in a partial Steiner triple system of order $2u + 1$ with a leave H such that either*

- $E(H) = E(L_1) \setminus E(L)$; *or*
- $E(H)$ *consists of* $E(L_1) \setminus E(L)$ *and the edges of a path of length three, with endvertices in* U *and internal vertices not in* U.

Proof. First note that since L is useable, either

(i) there are zero vertices with degree in L congruent to u (mod 2); or
(ii) there are two vertices with degree in L congruent to u (mod 2).

By Lemma 8.3, we can embed (U, A) in a partial Steiner triple system $(U \cup W, B)$ of order $u + w$ with a leave G where

- $w = |W| = u - 4$.
- $G(U) = L_1 - L$.
- $\deg_{G(U,W)}(x) \geqslant 1$ for every $x \in U$.
- $\epsilon(G(U, W)) = u + 2$.
- $\epsilon(G(W)) = u - 7$.

Clearly, $(U \cup W, B)$ satisfies the conditions of Lemma 8.6. Let $(U \cup W, B_1)$ be the repacking of $(U \cup W, B)$ obtained by applying Lemma 8.6 and let G_1 be its leave. Then $\chi'(G_1 - G_1(U)) = 3$, $G_1(U) = L_1 - L$, and $\deg_{G_1}(x) = 3$ for all $x \in W$. Furthermore, one of the following two conditions holds.

(i) There is a unique vertex $\alpha \in U$ with $\deg_{G_1 - G_1(U)}(\alpha) = 3$, and $\deg_{G_1 - G_1(U)}(x) = 1$ for all $x \in U \setminus \{\alpha\}$. (This occurs when no elements of U have degree in L congruent to u (mod 2).)
(ii) There are two vertices β and γ with $\deg_{G_1 - G_1(U)}(\beta) = \deg_{G_1 - G_1(U)}(\gamma) = 2$, and $\deg_{G_1 - G_1(U)}(x) = 1$ for all $x \in U \setminus \{\beta, \gamma\}$. (This occurs when there are precisely two elements of U, namely $\beta, \gamma \in U$, with degree in L congruent to u (mod 2).)

Let $W' = W \cup \{a_1, a_2, a_3\}$ where $a_1, a_2, a_3 \notin U \cup W$ and let ρ be a proper edge colouring of $G_1 - G_1(U)$ with colour set $\{a_1, a_2, a_3\}$. Let

$$B_2 = B_1 \cup \bigl\{\{x, y, \rho(xy)\}\colon xy \in E\bigl(G_1 - G_1(U)\bigr)\bigr\} \cup \bigl\{\{a_1, a_2, a_3\}\bigr\}.$$

Then $(U \cup W', B_2)$ is a partial Steiner triple system of order $2u - 1$ and we denote its leave by G_2. Notice that

- if a vertex has degree 1 in $G_1 - G_1(U)$ then it has degree 2 in $G_2 - G_2(U)$;
- if a vertex has degree 2 in $G_1 - G_1(U)$ then it has degree 1 in $G_2 - G_2(U)$; and
- if a vertex has degree 3 in $G_1 - G_1(U)$ then it has degree 0 in $G_2 - G_2(U)$.

In particular, $\deg_{G_2}(x) = 0$ for all $x \in W$, and $G_2(W')$ is empty. Let $(U \cup W', B_3)$ be the repacking of $(U \cup W', B_2)$ obtained by applying Lemma 7.2 and let G_3 be its leave. It is clear that either

(i) $G_3 - G_3(U)$ consists of cycles of even length (since it is bipartite with bipartition $\{U, W'\}$) and an isolated vertex, namely α; or
(ii) $G_3 - G_3(U)$ consists of cycles of even length and a path of even length with endvertices β and γ.

Let $W'' = W' \cup \{a_4, a_5\}$ where $a_4, a_5 \notin U \cup W'$, let ω be a proper edge colouring of $G_3 - G_3(U)$ with colour set $\{a_4, a_5\}$, and let

$$B_4 = B_3 \cup \big\{\{x, y, \omega(xy)\}\colon\ xy \in E\big(G_3 - G_3(U)\big)\big\}.$$

If $G_3 - G_3(U)$ consists of cycles of even length, then the leave G_4 of $(U \cup W'', B_4)$ has edge set $E(G_4) = (E(L_1) \setminus E(L)) \cup \{a_4\alpha, a_5\alpha, a_4a_5\}$, and we need only add the triple $\{a_4, a_5, \alpha\}$ to B_4 to obtain the required partial Steiner triple system with a leave H having edge set $E(H) = E(L_1) \setminus E(L)$.

If $G_3 - G_3(U)$ consists of cycles of even length and a path of even length with endvertices β and γ, then without loss of generality, we can assume that the edge incident with β in $G_3 - G_3(U)$ is assigned colour a_4 by ω and the edge incident with γ in $G_3 - G_3(U)$ is assigned colour a_5 by ω. It follows that the leave G_4 of $(U \cup W'', B_4)$ has edge set $E(G_4) = (E(L_1) \setminus E(L)) \cup \{a_5\beta, a_4\gamma, a_4a_5\}$, and $(U \cup W'', B_4)$ is the required partial Steiner triple system. □

Remark 1. Note that the vertex a_5 in the preceding proof satisfies one of the following two properties.

- a_5 is in precisely u U-type 1 triples (when $E(H) = E(L_1) \setminus E(L)$).
- a_5 is in precisely $u - 1$ U-type 1 triples (when $E(H)$ consists of $E(L_1) \setminus E(L)$ and the edges of a path of length three).

Remark 2. From the construction in the preceding proof, it is clear that if $E(H)$ consists of $E(L_1) \setminus E(L)$ and the edges of a path of length three, then the endvertices of the path are the two vertices with degree in L congruent to u (mod 2).

9. Partial Steiner triple systems with small leaves

In this section we remove the restriction $\epsilon(L) \geqslant 3u - 18$ from the main result of the preceding section by proving the corresponding result for $\epsilon(L) \leqslant 3u - 21$. We deal separately with the cases $\Delta(L) < \lceil \frac{u+1}{2} \rceil$ and $\Delta(L) \geqslant \lceil \frac{u+1}{2} \rceil$. In both cases we will need the following lemma.

Lemma 9.1. *Let u and w be integers with $w \leqslant u - 1$, let U and W be disjoint sets, with $|U| = u$ and $|W| = w$ and let $(U \cup W, B)$ be a partial Steiner triple system with a leave G satisfying the following conditions.*

(1) *$G(W)$ is empty.*
(2) $\deg_{G(U,W)}(x) = u + 1 - w$ *for all* $x \in W$.
(3) $\deg_{G(U,W)}(x) \leqslant u + 1 - w$ *for all* $x \in U$.
(4) *There are at most two elements of U with degree in $G(U, W)$ congruent to $u - w$* (mod 2).

Then $(U \cup W, B)$ can be embedded in a partial Steiner triple system of order $2u + 1$, with a leave H such that either

- *$E(H) = E(G(U))$; or*
- *$E(H)$ consists of $E(G(U))$ and the edges of a path of length three with endvertices in U and internal vertices not in $U \cup W$.*

Proof. The proof is by induction on $u + 1 - w$ (which is at least 2). If $u + 1 - w = 2$ then either

- $G(U, W)$ consists of cycles of even length and an isolated vertex, say α, in U; or
- $G(U, W)$ consists of cycles of even length and a path of even length with endvertices in U.

In both cases it is clear that $\chi'(G(U, W)) = 2$. Let $W' = W \cup \{a_1, a_2\}$ where $a_1, a_2 \notin U \cup W$, let ρ be a proper edge colouring of $G(U, W)$ with colour set $\{a_1, a_2\}$, and let

$$B' = B \cup \bigl\{\{x, y, \rho(xy)\}\colon xy \in E\bigl(G(U, W)\bigr)\bigr\}.$$

If $G(U, W)$ consists of cycles of even length, then the leave G' of $(U \cup W', B')$ has edge set $E(G') = E(G(U)) \cup \{a_1\alpha, a_2\alpha, a_1a_2\}$, and we need only add the triple $\{a_1, a_2, \alpha\}$ to B' to obtain the required partial Steiner triple system of order $2u + 1$.

Hence we can assume $G(U, W)$ consists of cycles of even length and a path of even length with endvertices in U. Let the endvertices of the path be β and γ. Without loss of generality we can assume that the edge incident with β in $G(U, W)$ is assigned colour a_1 by ρ, and the edge incident with γ in $G(U, W)$ is assigned colour a_2 by ρ. It follows that the leave G' of $(U \cup W', B')$ has edge set $E(G') = E(G(U)) \cup \{a_2\beta, a_1\gamma, a_1a_2\}$, and $(U \cup W', B')$ is the required partial Steiner triple system. Hence the lemma holds for $u + 1 - w = 2$.

Let $s > 2$, assume the lemma holds for $u + 1 - w < s$ and suppose $u + 1 - w = s$. Since $\Delta(G(U, W)) = s$ and $G(U, W)$ is bipartite, $\chi'(G(U, W)) = s$. Let ρ be a proper edge colouring of $G(U, W)$ with s colours. Note that there are exactly w edges of each colour, and that by Condition (4), there are at most two vertices of degree $s - 1$ in $G(U, W)$. Since $s > 2$, there is some colour, c say, such that there is an edge of colour c incident with each vertex of degree s or $s - 1$ in $G(U, W)$. Let $w' = w + 1$, $W' = W \cup \{a_3\}$ (where $a_3 \notin U \cup W$ so that $|W'| = w'$) and let

$$B' = B \cup \bigl\{\{a_3, x, y\}\colon xy \in E\bigl(G(U, W)\bigr),\ \rho(xy) = c\bigr\}.$$

Then $(U \cup W', B')$ is a partial Steiner triple system of order $u + w'$. Let G' be its leave. We show that $(U \cup W', B')$ satisfies the conditions of the lemma so that the result follows by the inductive hypothesis. Let $s' = u + 1 - w' = s - 1$.

Firstly, it is clear that $G'(W')$ is empty (since every vertex in W has degree s in $G(U, W)$) and that $\deg_{G'}(a_3) = u + w - 2w = s'$. Now note that for all $x \in U \cup W$, $\deg_{G'(U,W')}(x) = \deg_{G(U,W)}(x) - 1$ if x is incident with an edge of colour c in $G(U, W)$, and $\deg_{G'(U,W')}(x) = \deg_{G(U,W)}(x) + 1$ otherwise. It follows that $\deg_{G'(U,W')}(x) = s'$ for all $x \in W'$ and that $\deg_{G'(U,W')}(x) \leqslant s'$ for all $x \in U$. Finally, it is clear that $\deg_{G'(U,W')}(x) \equiv \deg_{G(U,W)}(x) + 1 \pmod 2$ and hence there are at most two vertices of U with degree in $G'(U, W)$ congruent to $s' - 1$. Since $u + 1 - w' = s' < s$, by the inductive hypothesis $(U \cup W', B')$ has an embedding in partial Steiner triple system of order $2u + 1$ with the required properties. This is also the required embedding of $(U \cup W, B)$. □

Remark. Note that the vertex a_2 in the preceding proof satisfies one of the following two properties.

- a_2 is in precisely u U-type 1 triples (when $(H) = E(G(U))$).
- a_2 is in precisely $u - 1$ U-type 1 triples (when $E(H)$ consists of $E(G(U))$ and the edges of a path of length three).

We now deal with the case of a partial Steiner triple system of order u having a leave containing a spanning useable subgraph L with $\epsilon(L) \leqslant 3u - 21$ and $\Delta(L) < \lceil \frac{u+1}{2} \rceil$.

Lemma 9.2. *Let (U, A) be a partial Steiner triple system of order $u \geqslant 11$, let L_1 be its leave, and suppose that L_1 contains a spanning useable subgraph L with $\epsilon(L) \leqslant 3u - 21$ and $\Delta(L) < \lceil \frac{u+1}{2} \rceil$. Then (U, A) can be embedded in a partial Steiner triple system of order $2u + 1$ with a leave H such that either*

- *$E(H) = E(L_1) \setminus E(L)$; or*
- *$E(H)$ consists of $E(L_1) \setminus E(L)$ and the edges of a path of length three with endvertices in U and internal vertices not in U.*

Proof. Let $w = \lceil \frac{u+1}{2} \rceil$ and let W be a set with $|W| = w$ and $W \cap U = \emptyset$. We shall embed (U, A) in a partial Steiner triple system $(U \cup W, B^*)$ with a leave G^* such that the required embedding of (U, A) can be obtained from $(U \cup W, B^*)$ by applying Lemma 9.1.

Since $\Delta(L) \leqslant w - 1$, there exists an equalised proper edge colouring γ of L with colour set W (see [11,17]). Label the elements of W by $a_1, a_2, \ldots, a_w$ so that if $i < j$ the number of edges assigned colour a_i by γ is at least the number of edges assigned colour a_j by γ. Let

$$B_0 = A \cup \bigl\{\bigl\{x, y, \gamma(xy)\bigr\}\colon\ xy \in E(L)\bigr\}$$

so that $(U \cup W, B_0)$ is a partial Steiner triple system.

It is straightforward to verify that for all $u \geqslant 11$ there exists a partial Steiner triple system of order w with $\frac{1}{3}\epsilon(L)$ triples. (For all n, the maximum number of triples in a partial Steiner triple system of order n is well known; see [14].) Moreover, by Lemma 7.2 there exists an equitable partial Steiner triple system (W, C) with $\frac{1}{3}\epsilon(L)$ triples and (by relabelling the elements if necessary) a leave M such that for $i < j$, $\deg_M(a_i) \leqslant \deg_M(a_j)$. Let $B_1 = B_0 \cup C$ so that $(U \cup W, B_1)$ is a partial Steiner triple system, and denote its leave by G_1.

It is easy to see that for each $x \in W$, the number of U-type 2 triples containing x equals the number of U-type 0 triples containing x. It follows that for each $x \in W$,

$$\deg_{G_1(U,W)}(x) = u - \bigl(w - 1 - \deg_{G_1(W)}(x)\bigr).$$

Note that $\epsilon(G_1(W)) = \binom{w}{2} - \epsilon(L) \geqslant \binom{w}{2} - 3u + 21$, which implies $\epsilon(G_1(W)) \geqslant 3$ (since $w = \lceil \frac{u+1}{2} \rceil$).

Recall that we wish to embed (U, A) in a partial Steiner triple system $(U \cup W, B^*)$ with a leave G^* which satisfies $\deg_{G^*(U,W)}(x) \leqslant u + 1 - w$ for all $x \in U$. To this end, we now create an intermediate system $(U \cup W, B_2)$ by adding certain U-type 1 triples to B_1, so that in the leave G_2 of $(U \cup W, B_2)$, $\deg_{G_2(U,W)}(x) \leqslant u + 1 - w$ for all $x \in U$.

Clearly $\deg_{G_1(U,W)}(x) \leqslant w$ for all $x \in U$. If u is odd then $u + 1 - w = w$, and so clearly $\deg_{G_1(U,W)}(x) \leqslant u + 1 - w$ for all $x \in U$. Hence, when u is odd we let $B_2 = B_1$. In the case u is even, $u + 1 - w = w - 1$, and it is possible that there exists a vertex $a \in U$ with $\deg_{G_1(U,W)}(a) > u + 1 - w$. If this is the case, then a satisfies $\deg_{G_1(U,W)}(a) = w$, and it follows that $\deg_L(a) = 0$. Hence, in the case u is even, for each $a \in U$ with $\deg_L(a) = 0$, we shall add a U-type 1 triple containing a. There are at most two such vertices (recall that L is useable). Thus, when u is even we construct B_2 from B_1 by arbitrarily adding a U-type 1 triple containing a for each $a \in U$ with $\deg_L(a) = 0$ (note that $G_1(W)$ has at least three edges). Let G_2 be the leave of the partial Steiner triple system $(U \cup W, B_2)$. Then $\deg_{G_2(U,W)}(x) \leqslant u + 1 - w$ for all $x \in U$.

We claim that we may now arbitrarily add further U-type 1 triples to B_2 until there are no unused edges with both vertices in W. To see this, suppose that we have added a (possibly empty) set S of U-type 1 triples to B_2, and denote the leave of the partial Steiner triple system $(U \cup W, B_2 \cup S)$ by G_3. First note that for each $x \in W$, each U-type 1 triple containing x uses one edge xa with $a \in U$ and one edge xb with $b \in W$. Hence, $\deg_{G_3(U,W)}(x) = \deg_{G_1(U,W)}(x) - r$, and $\deg_{G_3(W)}(x) = \deg_{G_1(W)}(x) - r$, where r is the number of U-type 1 triples containing x in $B_2 \cup S$. So we have

$$\deg_{G_3(U,W)}(x) = u - \bigl(w - 1 - \deg_{G_1(W)}(x)\bigr) - r = u - \bigl(w - 1 - \bigl(\deg_{G_1(W)}(x) - r\bigr)\bigr)$$

from which it follows that

$$\deg_{G_3(U,W)}(x) = u - \bigl(w - 1 - \deg_{G_3(W)}(x)\bigr).$$

We show that if $G_3(W)$ is not empty then we can add a U-type 1 triple to $B_2 \cup S$. Let $yz \in E(G_3(W))$. Then $\deg_{G_3(U,W)}(y) = u - (w - 1 - \deg_{G_3(W)}(y)) \geqslant u + 2 - w$, and similarly $\deg_{G_3(U,W)}(z) \geqslant u + 2 - w$. Since $2(u + 2 - w) > u$, y and z have (in G_3) a common neighbour in U, and hence we may add a U-type 1 triple to $B_2 \cup S$.

Let $(U \cup W, B^*)$ be the partial Steiner triple system which results from adding all the U-type 1 triples as described above, and let G^* be its leave. We claim that $(U \cup W, B^*)$ satisfies the conditions of Lemma 9.1. Clearly, $G^*(W)$ is empty, and this implies that $\deg_{G^*(U,W)}(x) = u - (w - 1 - \deg_{G^*(W)}(x)) = u + 1 - w$ for all $x \in W$. It is clear that $G^*(U) = L_1 - L$, and $\deg_{G^*(U,W)}(x) \leqslant u + 1 - w$ for all $x \in U$. Since L is useable, there are at most two vertices in U with degree in $G^*(U, W)$ congruent to $u - w \pmod 2$. Hence, we may apply Lemma 9.1 to obtain the required embedding of $(U \cup W, B^*)$, which is also an embedding of (U, A). □

Remark. From the remark following Lemma 9.1, it is clear that the system constructed in Lemma 9.2 also has a vertex $a_2 \notin U$ satisfying one of the two following properties:

- a_2 is in precisely u U-type 1 triples (when $E(H) = E(L_1) \setminus E(L)$).
- a_2 is in precisely $u - 1$ U-type 1 triples (when $E(H)$ consists of $E(L_1) \setminus E(L)$ and the edges of a path of length three).

We now consider the problem of embedding a partial Steiner triple system of order u with a leave L_1 containing a spanning useable subgraph L satisfying $\epsilon(L) \leqslant 3u - 21$ and $\Delta(L) \geqslant \lceil \frac{u+1}{2} \rceil$.

As in Lemma 9.2, we shall first embed the given partial Steiner triple system (U, A) of order u in a partial Steiner triple system of order $u + \lceil \frac{u+1}{2} \rceil$. Then, new vertices will be added one at a time, with new triples added after the addition of each new vertex, until we obtain a partial Steiner triple system which satisfies the conditions of Lemma 9.1. We shall then apply Lemma 9.1 to obtain the desired embedding.

At each stage we require that the partial Steiner triple system we have constructed thus far satisfies several conditions, and we include these conditions in Definition 9.3 below. Several of the conditions involve the value $\deg_{G(U,W)}(x) - \deg_{G(U)-(L_1-L)}(x)$ where $(U \cup W, B)$ is the constructed partial Steiner triple system, G is its leave, and $x \in U$. Thus, for brevity we introduce the notation $f_G(x)$ to denote this value. That is, given a partial Steiner triple system (U, A) with a leave L_1 containing a spanning useable subgraph L, and an embedding $(U \cup W, B)$ of (U, A) with a leave G, we define

$$f_G(x) = \deg_{G(U,W)}(x) - \deg_{G(U)-(L_1-L)}(x)$$

for all $x \in U$. Suppose $(U \cup W, B)$ and $(U \cup W, B \cup \{T\})$ are embeddings of (U, A) and let G and G' be their respective leaves. Note that if T is a U-type 2 triple such that T does not use an edge of $L_1 - L$, then for all $x \in U$, $f_{G'}(x) = f_G(x)$, whilst if T is a U-type 1 triple then $f_{G'}(x) = f_G(x) - 2$ for $x \in T \cap U$ and $f_{G'}(x) = f_G(x)$ for all $x \in U \setminus T$. Further, note that if $(U \cup W, B^*)$ with a leave G^* is a repacking of $(U \cup W, B)$, then $f_{G^*}(x) = f_G(x)$ for all $x \in U$. We will use these facts implicitly in the proofs that follow.

Definition 9.3. Suppose a partial Steiner triple system (U, A) of order u with a leave L_1 containing a spanning useable subgraph L is embedded in a partial Steiner triple system $(U \cup W, B)$ of order $u + w$ with a leave G, and let $L' = L_1 - L$. For a subset S of U, we say that such an embedding is S-good if the following conditions are satisfied.

(1) $\epsilon(L) \leqslant 3u - 21$.
(2) $\lceil \frac{u+1}{2} \rceil \leqslant w \leqslant u - 4$.
(3) $|S| \geqslant u - 5$.
(4) $G(U) = L_1 - L(S)$.
(5) $G(W)$ is empty.
(6) For all $x \in U \setminus S$, $\deg_{G(U,W)}(x) = w$.
(7) $\epsilon(G(U, W)) = w(u + 1 - w) + 2\epsilon(G(U) - L')$.
(8) For all $x \in S$, $f_G(x) \leqslant u + 1 - w$.
(9) For all $x \in S$, $1 \leqslant f_G(x)$.
(10) $|\{x \in S\colon\ f_G(x) = u + 1 - w \text{ or } f_G(x) = u - w\}| \leqslant w$.
(11) For all $x \in U \setminus S$, $\deg_{G(U)-L'}(x) \geqslant w - 1$.

Suppose an embedding of (U, A) in $(U \cup W, B)$ is S-good. It is straightforward to check that if $(U \cup W, B')$ is any repacking of $(U \cup W, B)$, then the embedding of (U, A) in $(U \cup W, B')$ is also S-good. This fact is used implicitly in the proofs that follow.

The following lemma is proven in [3], and is needed to prove Lemma 9.5.

Lemma 9.4. [3] *Let U and W be disjoint sets, let $(U \cup W, B)$ be a partial Steiner triple system, let G be its leave, and let $w = |W|$. Suppose there exists an edge $xy \in E(G(W))$ and a set Q of four vertices in U such that*

$$\sum_{x \in Q} \deg_{G(U,W)}(x) \geqslant 2w + 2.$$

Then there exists a partial Steiner triple system $(U \cup W, B')$ with a U-type 1 triple $T \in B'$ where one vertex of T is in Q and such that $(U \cup W, B' \setminus \{T\})$ is a repacking of $(U \cup W, B)$.

Lemma 9.5. *Let (U, A) be a partial Steiner triple system of order $u \geqslant 11$, let L_1 be its leave and suppose that L_1 contains a spanning useable subgraph L with $\epsilon(L) \leqslant 3u - 21$ and $\Delta(L) \geqslant \lceil \frac{u+1}{2} \rceil$. Then (U, A) has an S-good embedding in a partial Steiner triple system $(U \cup W, B)$ of order $u + w$ where $w = |W| = \lceil \frac{u+1}{2} \rceil$ and $S = \{x \in U\colon\ \deg_L(x) \leqslant w - 1\}$.*

Proof. It is straightforward to verify that for all $u \geqslant 11$, there exists a partial Steiner triple system (W, C) with $\frac{1}{3}\epsilon(L)$ triples. (For all n, the maximum number of triples in a partial Steiner triple system of order n is well known; see [14].)

Let $L' = L_1 - L$, let $s = |S|$, let $R = U \setminus S$ and let $r = |R| = u - s$. Note that $s \leqslant u - 1$ as $\Delta(L) \geqslant w$. The definition of S guarantees that $\Delta(L(S)) \leqslant w - 1$ and so by Vizing's Theorem [16] there exists a proper edge colouring γ of $L(S)$ with colour set W. Let

$$B_0 = A \cup C \cup \bigl\{\{x, y, \gamma(xy)\}\colon\ xy \in E\bigl(L(S)\bigr)\bigr\}.$$

Clearly $(U \cup W, B_0)$ is a partial Steiner triple system. Let G_0 be its leave and let $\varepsilon = \epsilon(G_0(W)) = \binom{w}{2} - \epsilon(L)$. Note that since $\epsilon(L) \leqslant 3u - 21$, we have $\varepsilon \geqslant 3$.

We now construct a sequence

$$(U \cup W, B_0), (U \cup W, B_1), \ldots, (U \cup W, B_\varepsilon)$$

of partial Steiner triple systems, with leaves $G_0, G_1, \ldots, G_\varepsilon$ say. For $i = 0, 1, 2, \ldots, \varepsilon - 1$, the partial Steiner triple system $(U \cup W, B_{i+1})$ is obtained from $(U \cup W, B_i)$ by one of the following two methods.

(A) If u is even and there exists a vertex $a \in U$ with $f_{G_i}(a) = w$, then we shall add a U-type 1 triple $\{a, x, y\}$ to B_i, for some $xy \in E(G_i(W))$.

(B) Otherwise we shall apply Lemma 9.4, where $Q = Q_i$ is defined to be a set consisting of four vertices of S such that $f_{G_i}(x) \geqslant f_{G_i}(y)$ for all $x \in Q_i$ and all $y \in S \setminus Q_i$.

Note that since both methods repack (possibly trivially) and add a new U-type 1 triple, $\epsilon(G_i(W)) = \varepsilon - i$ for $i = 0, 1, 2, \ldots, \varepsilon$. In particular, $G_\varepsilon(W)$ is empty.

We need to check that we can indeed construct this sequence. Suppose we have constructed $(U \cup W, B_k)$ for some $k \in \{0, 1, 2, \ldots, \varepsilon - 1\}$. We show that we can construct $(U \cup W, B_{k+1})$ by the appropriate method.

If u is even and there exists a vertex $a \in U$ with $f_{G_k}(a) = w$, then clearly $\deg_{G_k(U,W)}(a) = w$ and $\deg_L(a) = 0$. Since $k \leqslant \varepsilon - 1$, there is at least one edge, xy say, in $G_k(W)$. Let $B_{k+1} = B_k \cup \{\{a, x, y\}\}$.

Otherwise, we show that the conditions of Lemma 9.4 are satisfied with $Q = Q_k$ defined as above. Since $k \leqslant \varepsilon - 1$, there is at least one edge, xy say, in $G_k(W)$. We now check that $\sum_{x \in Q_k} \deg_{G_k(U,W)}(x) \geqslant 2w + 2$. Observe that $\epsilon(G_0(U, W)) = uw - 2\epsilon(L(S))$ from which it follows that $\epsilon(G_k(U, W)) = uw - 2\epsilon(L(S)) - 2k$. Since $k \leqslant \varepsilon - 1$ and $\varepsilon = \binom{w}{2} - \epsilon(L)$ we have

$$\begin{aligned}\epsilon\bigl(G_k(U, W)\bigr) &\geqslant uw - 2\epsilon\bigl(L(S)\bigr) - 2\left(\binom{w}{2} - \epsilon(L) - 1\right)\\ &= w(u + 1 - w) + 2\bigl(\epsilon(L) - \epsilon\bigl(L(S)\bigr)\bigr) + 2.\end{aligned}$$

Substituting

$$\epsilon\bigl(G_k(U, W)\bigr) = \sum_{x \in U} \deg_{G_k(U,W)}(x), \qquad \epsilon(L) - \epsilon\bigl(L(S)\bigr) = \frac{1}{2}\sum_{x \in U} \deg_{G_k(U)-L'}(x)$$

and

$$\sum_{x \in U} f_{G_k}(x) = \sum_{x \in U} \deg_{G_k(U,W)}(x) - \sum_{x \in U} \deg_{G_k(U)-L'}(x)$$

into the above inequality we obtain

$$\sum_{x\in U} f_{G_k}(x) \geqslant w(u+1-w)+2.$$

Now, for $x \in R$ we have $f_{G_k}(x) = w - \deg_{G_k(U)-L'}(x)$, which is at most zero since $\deg_{G_k(U)-L'}(x) \geqslant w$ for all $x \in R$. This implies that

$$\sum_{x\in S} f_{G_k}(x) \geqslant \sum_{x\in U} f_{G_k}(x)$$

and so we have

$$\sum_{x\in S} f_{G_k}(x) \geqslant w(u+1-w)+2.$$

It follows that

$$\sum_{x\in Q_k} f_{G_k}(x) \geqslant \frac{4}{s}\bigl(w(u+1-w)+2\bigr).$$

It is straightforward to check that since $w = \lceil \frac{u+1}{2} \rceil$,

$$4\bigl(w(u+1-w)+2\bigr) > (u-1)(2w+1).$$

We have $s \leqslant u-1$ and so it follows that

$$\sum_{x\in Q_k} f_{G_k}(x) > 2w+1 \quad \text{and hence that} \quad \sum_{x\in Q_k} f_{G_k}(x) \geqslant 2w+2.$$

Since $f_{G_k}(x) = \deg_{G_k(U,W)}(x) - \deg_{G_k(U)-L'}(x)$, this implies

$$\sum_{x\in Q_k} \deg_{G_k(U,W)}(x) \geqslant 2w+2$$

as required. So we can indeed construct the above sequence of partial Steiner triple systems.

It remains only to verify that $(U \cup W, B_\varepsilon)$ is S-good. Conditions (1), (2), (4)–(6), and (11) of Definition 9.3 are clearly satisfied. We now verify that the remaining conditions are also satisfied. At this point we also note that for all $x \in S$, $f_{G_0}(x) = w - \deg_L(x)$. Thus, it follows from the construction of $(U \cup W, B_\varepsilon)$, and the properties of repacking, that for all $x \in S$ we have $f_{G_\varepsilon}(x) \leqslant w - \deg_L(x)$ and $f_{G_\varepsilon}(x) < w - \deg_L(x)$ if and only if x occurs in a U-type 1 triple in B_ε. This can happen only if u is even and $f_{G_0}(x) = w$, or if $x \in Q_i$ for some $i \in \{0, 1, 2, \dots, \varepsilon - 1\}$.

We now check Conditions (3), (7)–(10) of Definition 9.3.

Condition (3). We first establish a weak upper bound on r. Since $\deg_L(x) \geqslant \lceil \frac{u+1}{2} \rceil$ for each $x \in R$, and

$$\sum_{x\in R} \deg_L(x) \leqslant \sum_{x\in U} \deg_L(x),$$

we have

$$r\left\lceil \frac{u+1}{2} \right\rceil \leqslant \sum_{x\in U} \deg_L(x).$$

Furthermore,

$$\sum_{x\in U}\deg_L(x)=2\epsilon(L),$$

and $\epsilon(L)\leqslant 3u-21$. This implies that

$$r\left\lceil\frac{u+1}{2}\right\rceil\leqslant 2(3u-21).$$

It follows that $r\leqslant\frac{12u-84}{u+1}$, and hence $r\leqslant 11$. Also note that $r\leqslant 5$ for all $u\leqslant 14$.

We will now show that $r\leqslant 5$ for all $u\geqslant 15$. Simple counting yields the following equation:

$$\sum_{x\in R}\deg_L(x)=2\epsilon\big(L(R)\big)+\epsilon\big(L(R,S)\big).$$

Since $\epsilon(L(R))\leqslant\binom{r}{2}$, $\epsilon(L(R))+\epsilon(L(R,S))\leqslant\epsilon(L)$ and $\epsilon(L)\leqslant 3u-21$, we have

$$r\left\lceil\frac{u+1}{2}\right\rceil\leqslant\sum_{x\in R}\deg_L(x)\leqslant 3u-21+\frac{r(r-1)}{2}.$$

Hence, $3u-21+\frac{r(r-1)}{2}-r\frac{u+1}{2}\geqslant 0$. It is straightforward to verify that this inequality fails for $r\in\{6,7,\dots,11\}$ and $u\geqslant 15$. Thus $r\leqslant 5$ for all $u\geqslant 11$ and Condition (3) holds.

Condition (7). Clearly $\epsilon(G_\varepsilon(U,W))=uw-2\epsilon(L(S))-2\varepsilon$. Substituting $\varepsilon=\binom{w}{2}-\epsilon(L)$, we obtain

$$\begin{aligned}\epsilon\big(G_\varepsilon(U,W)\big)&=uw-2\epsilon\big(L(S)\big)-2\left(\binom{w}{2}-\epsilon(L)\right)\\&=w(u+1-w)+2\big(\epsilon(L)-\epsilon\big(L(S)\big)\big).\end{aligned}$$

Since $\epsilon(G_\varepsilon(U)-L')=\epsilon(L)-\epsilon(L(S))$, we have Condition (7).

Condition (8). Let $x\in S$. We have seen that $f_{G_\varepsilon}(x)\leqslant w-\deg_L(x)$. If u is odd then $u+1-w=w$ and Condition (8) holds. If u is even then $u+1-w=w-1$. In this case, if $\deg_L(x)\geqslant 1$ we are finished, so assume that $\deg_L(x)=0$. Since L is useable and u is even, there are at most two vertices in U with degree 0 in L. Since $\varepsilon\geqslant 3$, x is in at least one U-type 1 triple (by applying method (A) to obtain B_1 from B_0, or B_2 from B_1), and so $f_{G_\varepsilon}(x)\leqslant w-2\leqslant u+1-w$ as required.

Condition (9). We have already seen that for $x\in S$, $f_{G_0}(x)=w-\deg_L(x)$. Since $\deg_L(x)\leqslant w-1$ for all $x\in S$, we have $f_{G_0}(x)\geqslant 1$ for all $x\in S$. To show that $f_{G_\varepsilon}(x)\geqslant 1$ for all $x\in S$, we must show that for each $i=0,1,2,\dots,\varepsilon-1$, if $\{a,b,c\}$ is a U-type 1 triple in B_{i+1} with $a\in U$, such that $(U\cup W, B_{i+1}\setminus\{\{a,b,c\}\})$ is a repacking of $(U\cup W, B_i)$, then $f_{G_i}(a)\geqslant 3$. Recall that B_{i+1} is obtained from B_i by method (A) or (B).

If B_{i+1} is obtained from B_i by method (A), then u is even and $f_{G_i}(x)=w$ for some $x\in S$. Clearly, $f_{G_i}(x)\geqslant 3$, since $u\geqslant 11$ and $w=\lceil\frac{u+1}{2}\rceil$.

Otherwise, B_{i+1} is obtained from B_i by method (B), and it is sufficient to show that $f_{G_i}(x)\geqslant 3$ for all $x\in Q_i$. Suppose otherwise. That is, suppose there exists an element $a\in Q_i$

with $f_{G_i}(a) \leqslant 2$. By the definition of Q_i, this implies that $f_{G_i}(x) \leqslant 2$ for all $x \in S \setminus Q_i$. We also know that $f_{G_i}(x) \leqslant u+1-w$ for all $x \in Q_i$. Hence we have

$$\sum_{x\in S} f_{G_i}(x) \leqslant 3(u+1-w)+2(s-3).$$

But we have already seen that

$$\sum_{x\in S} f_{G_i}(x) \geqslant w(u+1-w)+2$$

so it must be the case that

$$w(u+1-w)+2 \leqslant 3(u+1-w)+2(s-3).$$

Substituting $s \leqslant u-1$ into this expression and simplifying we get

$$(w-3)(u+1-w)-2(u-5) \leqslant 0.$$

Substituting $w = \frac{u+1}{2}$ for u odd and $w = \frac{u+2}{2}$ for u even into this expression we get

$$\frac{1}{4}\left(u^2-12u+35\right) \leqslant 0 \quad \text{and} \quad \frac{1}{4}\left(u^2-12u+40\right) \leqslant 0$$

for u odd and u even, respectively. Since $u \geqslant 11$, this is a contradiction and Condition (9) holds.

Condition (10). Consider first the case u is odd. Then $w = u+1-w = \frac{u+1}{2}$. Let $x \in S$. If $f_{G_\varepsilon}(x) = u+1-w$, then clearly $\deg_L(x) = 0$. But $\Delta(L) \geqslant \frac{u+1}{2}$ so there are at most $u-1-\frac{u+1}{2} = \frac{u-3}{2}$ vertices of S having degree 0 in L. Since L is useable, there are at most two vertices with $f_{G_\varepsilon}(x) \equiv u-w \pmod 2$. Thus

$$\left|\left\{x \in S\colon\ f_{G_\varepsilon}(x) = u+1-w \text{ or } f_{G_\varepsilon}(x) = u-w\right\}\right| \leqslant \frac{u-3}{2}+2$$

and Condition (10) holds when u is odd.

Now consider the case u is even. Then $u+1-w = w-1$. We have noted that $f_{G_\varepsilon}(x) < w-\deg_L(x)$ whenever x is in a U-type 1 triple, and that there are at most two vertices with $f_{G_\varepsilon}(x) \equiv u-w \pmod 2$. If there exists an $x \in Q_{\varepsilon-1}$ such that $f_{G_{\varepsilon-1}}(x) < u+1-w$ then clearly there are at most five vertices of S with $f_{G_\varepsilon}(x) = u+1-w$ or $f_{G_\varepsilon}(x) = u-w$ and we are finished (since $u \geqslant 11$ implies $w \geqslant 6$).

Hence we can assume that for all $x \in Q_{\varepsilon-1}$, we have $f_{G_{\varepsilon-1}}(x) = u+1-w$. By the definition of Q_i for $i = 0,1,2,\dots,\varepsilon-1$, this implies that every U-type 1 triple in B_ε is incident with a distinct vertex x where $f_{G_\varepsilon}(x) = u-w$ or $f_{G_\varepsilon}(x) = u-w-1$.

We know there are ε U-type 1 triples in B_ε. Since L is useable, there are at most two vertices with $f_{G_\varepsilon}(x) = u-w$, and at most two U-type 1 triples added by method (A). So if $s-(\varepsilon-2) \leqslant w$ then $|\{x \in S\colon\ f_{G_\varepsilon}(x) = u+1-w \text{ or } f_{G_\varepsilon}(x) = u-w\}| \leqslant w$. Hence we need only show that $\varepsilon - s + w - 2 \geqslant 0$. Since u is even we have $w = \frac{u+2}{2}$ and we know that $s \leqslant u-1$, $\varepsilon = \binom{w}{2} - \epsilon(L)$, and $\epsilon(L) \leqslant 3u-21$. Hence we have

$$\varepsilon - s + w - 2 \geqslant \binom{\frac{1}{2}(u+2)}{2} - (3u-21) - (u-1) + \frac{u+2}{2} - 2.$$

Simplifying we get $\varepsilon - s + w - 2 \geqslant \frac{1}{8}(u^2 - 26u + 168)$. It is straightforward to check that this is non-negative for every even integer u. $\square$

We now prove results that allow us to extend the S-good embedding given by Lemma 9.5 to an S-good embedding in a partial Steiner triple system of larger order, or to an S'-good embedding, where S is a proper subset of S'. These results are given in Lemmas 9.7 and 9.8, respectively. The proofs of these lemmas use the following repacking result, proven in [3].

Lemma 9.6. [3] *Let U and W be disjoint sets, let $(U \cup W, B)$ be a partial Steiner triple system, let G be its leave, let $w = |W|$ and suppose $\deg_{G(U,W)}(x) \geqslant 1$ for all $x \in U$. Let $P = \{x_1, x_2, \dots, x_p\}$ be any subset of U with $p \leqslant w$ and $\deg_{G(U,W)}(x) \geqslant 2$ for all $x \in P$. Then there exists a repacking $(U \cup W, B')$ of $(U \cup W, B)$ with a leave G' such that G' has a matching $\{x_1y_1, x_2y_2, \dots, x_py_p\}$ for some $y_1, y_2, \dots, y_p \in W$.*

Lemma 9.7. *Let (U, A) be a partial Steiner triple system of order $u \geqslant 11$, let L_1 be its leave, suppose that L_1 contains a spanning useable subgraph L with $\epsilon(L) \leqslant 3u - 21$, and suppose that for some $S \subseteq U$, (U, A) has an S-good embedding in a partial Steiner triple system of order $u + w$ with a leave G. If $S \neq U$ and for all $x \in U \setminus S$, $\deg_{G(U)-(L_1-L)}(x) \geqslant w$, then (U, A) has an S-good embedding in a partial Steiner triple system of order $u + w + 1$.*

Proof. Let $L' = L_1 - L$ and let W be a set with $|W| = w$ and $U \cap W = \emptyset$. Suppose that (U, A) has an S-good embedding in a partial Steiner triple system $(U \cup W, B)$ with a leave G where $S \neq U$ and $\deg_{G(U)-L'}(x) \geqslant w$ for all $x \in U \setminus S$. Note that $G(U) - L' = L - L(S)$ (see Condition (4) of Definition 9.3). Let $R = U \setminus S$ and let $r = |R|$.

First we show that $|S| \geqslant w$. Note that $w \leqslant u - 4$ and $|S| \geqslant u - 5$ (see Conditions (2) and (3) of Definition 9.3). Thus, $|S| \geqslant w$ unless $|S| = u - 5$ and $w = u - 4$. In this case, note that $r = 5$, and $\epsilon(L(R)) \leqslant 10$. Since $\deg_{G(U)-L'}(x) \geqslant u - 4$ for all $x \in R$, we have

$$\sum_{x \in R} \deg_{G(U)-L'}(x) \geqslant 5(u - 4).$$

Furthermore, $\epsilon(L(R)) + \epsilon(L(R, S)) \leqslant \epsilon(L) \leqslant 3u - 21$. Hence, it follows from

$$\sum_{x \in R} \deg_{G(U)-L'}(x) = 2\epsilon\big(L(R)\big) + \epsilon\big(L(R, S)\big)$$

that $5(u - 4) \leqslant 3u - 21 + 10$, which is a contradiction, since $u \geqslant 11$.

Let P be a subset of S with $|P| = w$ such that for all $x \in P$ and all $y \in S \setminus P$, $f_G(x) \geqslant f_G(y)$. We denote the elements of P by $x_1, x_2, \dots, x_w$.

Conditions (6) and (9) imply that $\deg_{G(U,W)}(x) \geqslant 1$ for all $x \in U$. So if $\deg_{G(U,W)}(x) \geqslant 2$ for all $x \in P$ then we can apply Lemma 9.6. We now show that this is indeed the case. Suppose otherwise, and let $x \in P$ with $f_G(x) \leqslant 1$ (recall that $f_G(x) \leqslant \deg_{G(U,W)}(x)$). We know that $1 \leqslant f_G(x) \leqslant u + 1 - w$ for all $x \in S$ and so by the definition of P we have

$$\sum_{x \in S} f_G(x) \leqslant (w - 1)(u + 1 - w) + \big(|S| - (w - 1)\big).$$

Now, $|S| \leqslant u - 1$. Replacing $|S|$ with $u - 1$ in the above expression and simplifying we obtain

$$\sum_{x \in S} f_G(x) \leqslant w(u + 1 - w) - 1.$$

Notice now that for $x \in U \setminus S$ we have $f_G(x) = w - \deg_{G(U)-L'}(x)$, which is at most zero since $\deg_{G(U)-L'}(x) \geqslant w$ for all $x \in U \setminus S$. This implies that

$$\sum_{x \in S} f_G(x) \geqslant \sum_{x \in U} f_G(x).$$

But

$$\sum_{x \in U} f_G(x) = \sum_{x \in U} \deg_{G(U,W)}(x) - \sum_{x \in U} \deg_{G(U)-L'}(x),$$
$$\sum_{x \in U} \deg_{G(U,W)}(x) = \epsilon\big(G(U, W)\big)$$

and

$$\sum_{x \in U} \deg_{G(U)-L'}(x) = 2\epsilon\big(G(U) - L'\big).$$

So we have

$$\sum_{x \in S} f_G(x) \geqslant \epsilon\big(G(U, W)\big) - 2\epsilon\big(G(U) - L'\big) = w(u + 1 - w)$$

(see Condition (7) of Definition 9.3). This is a contradiction, so $\deg_{G(U,W)}(x) \geqslant 2$ for all $x \in P$.

Thus we can apply Lemma 9.6 to obtain a repacking $(U \cup W, B^*)$ of $(U \cup W, B)$ with a leave G^* such that G^* has a matching $\{x_1y_1, x_2y_2, \ldots, x_wy_w\}$ with $y_1, y_2, \ldots, y_w \in W$. Choose a new element $\beta \notin U \cup W$, let $W' = W \cup \{\beta\}$, let $w' = w + 1$, let

$$B' = B^* \cup \big\{\{\beta, x_1, y_1\}, \{\beta, x_2, y_2\}, \ldots, \{\beta, x_w, y_w\}\big\}$$

and let G' be the leave of $(U \cup W', B')$. Note that, for all $x \in P$ we have $f_{G'}(x) = f_G(x) - 1$ and for all $x \in U \setminus P$ we have $f_{G'}(x) = f_G(x) + 1$.

We claim that the embedding of (U, A) in $(U \cup W', B')$ is S-good. It is either clear or routine to check that Conditions (1), (3)–(6) and (11) of Definition 9.3 are satisfied. To verify Condition (11) we use the assumption that $\deg_{G(U)-L'}(x) \geqslant w$ for all $x \in U \setminus S$. We now verify that the remaining conditions are also satisfied.

Condition (2). Since $S \neq U$, $\deg_{G(U)-L'}(x) \geqslant w$ for all $x \in U \setminus S$, and $\Delta(G(U) - L') \leqslant u - 5$ (since L is useable), we have $w \leqslant u - 5$ which implies $w' \leqslant u - 4$ as required.

Condition (7). We have $\epsilon(G'(U) - L') = \epsilon(G(U) - L')$ and $\epsilon(G'(U, W)) = \epsilon(G(U, W)) + u - 2w$. Hence it follows from $\epsilon(G(U, W)) = w(u + 1 - w) + 2\epsilon(G(U) - L')$ that

$$\epsilon\big(G'(U, W)\big) = w(u + 1 - w) + 2\epsilon\big(G'(U) - L'\big) + u - 2w.$$

Replacing w with $w' - 1$ in this expression and simplifying we get

$$\epsilon\big(G'(U, W)\big) = w'(u + 1 - w') + 2\epsilon\big(G'(U) - L'\big)$$

and Condition (7) holds.

Condition (8). If $x \in P$, we have $f_{G'}(x) = f_G(x) - 1$. Since $f_G(x) \leqslant u + 1 - w$ (see Condition (8) of Definition 9.3), we have $f_{G'}(x) \leqslant u + 1 - w - 1 = u + 1 - w'$ as required.

Now let $x \in S \setminus P$. By the definition of P and by Condition (10) of Definition 9.3 we have $f_G(x) \leqslant u - w - 1$ and hence $f_{G'}(x) = f_G(x) + 1 \leqslant u - w = u + 1 - w'$ as required.

Condition (9). We have already proven that $f_G(x) \geqslant 2$ for all $x \in P$. Since $f_{G'}(x) = f_G(x) - 1$ for $x \in P$ and $f_{G'}(x) = f_G(x) + 1$ for $x \in U \setminus P$, Condition (9) holds.

Condition (10). Let $Q = \{x \in S\colon\ f_{G'}(x) = u + 1 - w' \text{ or } f_{G'}(x) = u - w'\}$. Clearly, $Q = \{x \in P\colon\ f_G(x) = u + 1 - w \text{ or } f_G(x) = u - w\} \cup \{x \in S \setminus P\colon\ f_G(x) = u - w - 1 \text{ or } f_G(x) = u - w - 2\}$.

Suppose there is a vertex $x \in P$ with $f_G(x) = u - w - 2$. Then $|\{x \in P\colon\ f_G(x) = u + 1 - w \text{ or } f_G(x) = u - w\}| \leqslant w - 1$. Since L is useable, $|\{x \in U\colon\ f_G(x) \equiv u - w \pmod 2\}| \leqslant 2$. This, together with the definition of P, implies that $|\{x \in S \setminus P\colon\ f_G(x) = u - w - 1 \text{ or } f_G(x) = u - w - 2\}| \leqslant 1$. Hence, $|Q| \leqslant w \leqslant w'$ and we are finished.

Similarly, if there is a vertex $x \in P$ with $f_G(x) \leqslant u - w - 3$ then $|Q| \leqslant w - 1$ and we are finished. Hence it remains only to consider the case where $f_G(x) \geqslant u - w - 1$ for all $x \in P$. In this case we suppose that Q has at least $w' + 1 = w + 2$ elements, and show that this leads to a contradiction.

We first obtain a lower bound on $\sum_{x\in S} f_G(x)$. Let $t = |\{x \in P\colon\ f_G(x) = u - w - 1\}|$. Since $|Q| \geqslant w + 2$ and P is chosen such that $f_G(x) \geqslant f_G(y)$ for all $x \in P$ and all $y \in S \setminus P$, we have

- $|\{x \in P\colon\ f_G(x) = u + 1 - w \text{ or } f_G(x) = u - w\}| = w - t$;
- $|\{x \in P\colon\ f_G(x) = u - w - 1\}| = t$;
- $|\{x \in S \setminus P\colon\ f_G(x) = u - w - 1 \text{ or } f_G(x) = u - w - 2\}| \geqslant t + 2$; and
- $f_G(x) \geqslant 1$ for all $x \in S$ (by Condition (9) of Definition 9.3).

Furthermore, since L is useable, $|\{x \in U : f_G(x) \equiv u - w \pmod 2\}| \leqslant 2$. Hence we have

$$\sum_{x\in S} f_G(x) \geqslant (w-t)(u+1-w) + t(u-w-1) + (t+2)(u-w-1) \\ - 2 + \bigl(|S| - (w+t+2)\bigr)$$

which simplifies to

$$\sum_{x\in S} f_G(x) \geqslant t(u-w-4) + w(u+1-w) + 2u - 3w - 6 + |S|.$$

Now, consider the term $t(u - w - 4)$. Since $S \neq U$, $w + t + 2 \leqslant u - 1$, which implies $u - w \geqslant t + 3$. So $t(u - w - 4) \geqslant t(t - 1) \geqslant 0$ (since $t \geqslant 0$). Hence we have

$$\sum_{x\in S} f_G(x) \geqslant w(u+1-w) + 2u - 3w - 6 + |S|.$$

We now derive an upper bound on $\sum_{x\in S} f_G(x)$. Since $G(S) - L'$ is empty (see Condition (4) of Definition 9.3), we have

$$\sum_{x\in S} \deg_{G(U)-L'}(x) = \epsilon\bigl(L(R,S)\bigr) = \epsilon\bigl(G(U) - L'\bigr) - \epsilon\bigl(L(R)\bigr).$$

But $\epsilon(L(R)) \leqslant \binom{r}{2}$ and so

$$\sum_{x\in S} \deg_{G(U)-L'}(x) \geqslant \epsilon\bigl(G(U) - L'\bigr) - \frac{r(r-1)}{2}. \qquad (*)$$

Consider now the equation

$$\sum_{x\in S} f_G(x) = \sum_{x\in S}\deg_{G(U,W)}(x) - \sum_{x\in S}\deg_{G(U)-L'}(x).$$

Combining

$$\sum_{x\in S}\deg_{G(U,W)}(x) = \epsilon\big(G(S,W)\big) = \epsilon\big(G(U,W)\big) - \epsilon\big(G(R,W)\big)$$

with $(*)$ derived above, we have

$$\sum_{x\in S} f_G(x) \leqslant \epsilon\big(G(U,W)\big) - \epsilon\big(G(R,W)\big) - \epsilon\big(G(U)-L'\big) + \frac{r(r-1)}{2}.$$

Furthermore, $\epsilon(G(R,W)) = wr$ and $\epsilon(G(U,W)) = w(u+1-w) + 2\epsilon(G(U)-L')$ (see Conditions (6) and (7) of Definition 9.3), and hence

$$\sum_{x\in S} f_G(x) \leqslant w(u+1-w) + \epsilon\big(G(U)-L'\big) - wr + \frac{r(r-1)}{2}.$$

Combining this with the lower bound for $\sum_{x\in S} f_G(x)$ obtained above, we have

$$w(u+1-w) + 2u - 3w - 6 + |S| \leqslant w(u+1-w) + \epsilon\big(G(U)-L'\big) - wr + \frac{r(r-1)}{2}$$

which simplifies to

$$2u - 3w - 6 + |S| \leqslant \epsilon\big(G(U)-L'\big) - wr + \frac{r(r-1)}{2}.$$

Since $\epsilon(G(U)-L') \leqslant 3u-21$ and $|S| \geqslant u-5$ we have

$$\frac{r(r-1)}{2} - w(r-3) - 10 \geqslant 0$$

which is a contradiction for $r \geqslant 3$ (recall that $r \leqslant 5$). We now show that we also obtain a contradiction for $r \in \{1,2\}$. We know that $\Delta(L) \leqslant u-5$ (since L is useable) and $L(S)$ is empty (see Condition (4) of Definition 9.3), and so it is clear that $\epsilon(G(U)-L') \leqslant r(u-5)$. Furthermore, $|S| = u-r$. Thus, it follows from the inequality

$$2u - 3w - 6 + |S| \leqslant \epsilon\big(G(U)-L'\big) - wr + \frac{r(r-1)}{2}$$

derived above that

$$(3-r)(u-w-2) + 2r - \frac{r(r-1)}{2} \leqslant 0.$$

Since $r \in \{1,2\}$ and $u-w-2 \geqslant 2$ (see Condition (2) of Definition 9.3) this is also a contradiction. Hence Condition (10) holds. □

Lemma 9.8. *Let (U,A) be a partial Steiner triple system of order u and let L_1 be its leave. Suppose that L_1 contains a spanning useable subgraph L and suppose that for some $S \subseteq U$, (U,A) has an S-good embedding in a partial Steiner triple system of order $u+w$ with a leave G. If there exists a vertex $\alpha \in U \setminus S$ such that $\deg_{G(U)-(L_1-L)}(\alpha) = w-1$, then (U,A) has an $(S\cup\{\alpha\})$-good embedding in a partial Steiner triple system of order $u+w$.*

Proof. Let $L' = L_1 - L$, let W be a set with $|W| = w$ and $W \cap U = \emptyset$. Let $(U \cup W, B)$ be a partial Steiner triple system in which (U, A) has an S-good embedding, let G be the leave of $(U \cup W, B)$, and let $\alpha \in U \setminus S$ with $\deg_{G(U)-L'}(\alpha) = w - 1$. Let $P = \mathrm{Nbd}_{G(U)-L'}(\alpha) \cap S = \{x_1, x_2, \ldots, x_t\}$. Then $t \leqslant w - 1$.

Since $\deg_{G(U)-L'}(x) \geqslant 1$ for all $x \in P$, Conditions (6) and (9) of Definition 9.3 imply that $\deg_{G(U,W)}(x) \geqslant 2$ for all $x \in P$ and that $\deg_{G(U,W)}(x) \geqslant 1$ for all $x \in U \setminus P$. Hence we can apply Lemma 9.6 and thus obtain a repacking $(U \cup W, B^*)$ of $(U \cup W, B)$ with a leave G^* such that G^* has a matching $\{x_1y_1, x_2y_2, \ldots, x_ty_t\}$ for some $y_1, y_2, \ldots, y_t \in W$. Let

$$B' = B^* \cup \bigl\{\{\alpha, x_1, y_1\}, \{\alpha, x_2, y_2\}, \ldots, \{\alpha, x_t, y_t\}\bigr\}$$

and let G' be the leave of $(U \cup W, B')$. Note that for all $x \in U$, $f_{G'}(x) = f_G(x)$. It is routine to check that the embedding of (U, A) in $(U \cup W, B')$ is $(S \cup \{\alpha\})$-good. □

We are now ready to prove the main result of this section.

Lemma 9.9. *Let (U, A) be a partial Steiner triple system of order $u \geqslant 11$, let L_1 be its leave and suppose that L_1 contains a spanning useable subgraph L with $\epsilon(L) \leqslant 3u - 21$. Then (U, A) can be embedded in a partial Steiner triple system of order $2u + 1$ with a leave H such that either*

- *$E(H) = E(L_1) \setminus E(L)$; or*
- *$E(H)$ consists of $E(L_1) \setminus E(L)$ and the edges of a path of length three with endvertices in U and internal vertices not in U.*

Proof. Let $L' = L_1 - L$. If $\Delta(L) < \lceil \frac{u+1}{2} \rceil$ then the result follows by Lemma 9.2. So we can assume $\Delta(L) \geqslant \lceil \frac{u+1}{2} \rceil$. Hence by Lemma 9.5, for some $S_0 \subseteq U$, (U, A) has an S_0-good embedding in a partial Steiner triple system $(U \cup W_0, B_0)$ of order $u + w_0$ where $w_0 = |W_0| = \lceil \frac{u+1}{2} \rceil$. Let G_0 be the leave of $(U \cup W_0, B_0)$.

We now repeatedly apply Lemma 9.7 or Lemma 9.8, at each stage producing a new partial Steiner triple system $(U \cup W, B)$ in which (U, A) has an S-good embedding for some $S \subseteq U$, until we obtain a U-good embedding of (U, A). We need to specify which of these lemmas we apply at each stage, verify that we can indeed apply them, and verify that we eventually obtain a U-good embedding of (U, A).

Suppose we have an S-good embedding, with $S \neq U$, of (U, A) in a partial Steiner triple system $(U \cup W, B)$. Let G be the leave of $(U \cup W, B)$ and let $w = |W|$. We apply Lemma 9.7 in the case $\deg_{G(U)-L'}(x) \geqslant w$ for all $x \in U \setminus S$, and we apply Lemma 9.8 if there exists an $\alpha \in U \setminus S$ such that $\deg_{G(U)-L'}(\alpha) = w - 1$. Recall that Condition (11) of Definition 9.3 guarantees $\deg_{G(U)}(x) \geqslant w - 1$ for all $x \in U \setminus S$. It is thus easy to see that for $S \neq U$, we can indeed apply either Lemma 9.7 or Lemma 9.8. Since Lemma 9.7 results in an S-good embedding of (U, A) in a partial Steiner triple system of order $u + w + 1$, and Lemma 9.8 results in an $(S \cup \{\alpha\})$-good embedding of (U, A) in a partial Steiner triple system of order $u + w$, it is clear that we eventually obtain a U-good embedding of (U, A). Note that Condition (2) of Definition 9.3 guarantees that the order of the resulting partial Steiner triple system is at most $2u - 4$.

Hence we can indeed construct a U-good embedding $(U \cup W', B')$ of (U, A). Let G' be the leave of $(U \cup W', B')$ and let $w' = |W'|$. Applying Lemma 7.2 we obtain a repacking $(U \cup W', B^*)$ of $(U \cup W', B')$ with a leave G^* such that $|\deg_{G^*}(x) - \deg_{G^*}(y)| \leqslant 2$ for all $x, y \in W'$. We now show that $(U \cup W', B^*)$ satisfies the conditions of Lemma 9.1.

By Conditions (4) and (5) of Definition 9.3, $G^*(U) = L'$ and $G^*(W')$ is empty. By Condition (7) of Definition 9.3 we have $\epsilon(G^*(U, W)) = w'(u + 1 - w')$ which implies that $\deg_{G^*(U,W)}(x) = \deg_{G^*}(x) = u + 1 - w'$ for all $x \in W'$ (since $\deg_{G^*}(x)$ has opposite parity to $u + w'$ for all $x \in W'$). Condition (8) of Definition 9.3 guarantees that $\deg_{G^*(U,W)}(x) \leqslant u + 1 - w'$ for all $x \in U$. Hence the result follows by Lemma 9.1. □

Remark 1. From the remarks following Lemmas 9.1 and 9.2, it is clear that the system constructed in Lemma 9.9 also has a vertex $a_2 \notin U$ satisfying one of the following two properties:

- a_2 is in precisely u U-type 1 triples (when $E(H) = E(L_1) \setminus E(L)$).
- a_2 is in precisely $u - 1$ U-type 1 triples (when $E(H)$ consists of $E(L_1) \setminus E(L)$ and the edges of a path of length three).

Remark 2. From the construction in the preceding proof, it is clear that if $E(H)$ consists of $E(L_1) \setminus E(L)$ and the edges of a path of length three, then the endvertices of the path are the two vertices with degree in L congruent to u (mod 2).

10. Main result

Before proving the main result of this paper, we require the following lemma which is used in the final step of our embedding method.

Lemma 10.1. *Let $U_0 \subset U_1$ and suppose that (U_0, A_0) is a partial extended triple system which is embedded in a partial extended triple system $(U_1 \cup W, B)$ of odd order. Let L be the leave of $(U_1 \cup W, B)$ and suppose that one of the following statements holds.*

- *There exists a vertex $k \in W$ such that each extended triple in B containing k is a normal triple which contains at least one other vertex in W; and $E^+(L) = \{xx\colon x \in W\}$.*
- *There exists a vertex $k \in U_1 \setminus U_0$ such that each extended triple in B containing k is either $\{k, k, k\}$ or a normal triple which contains at least one vertex in W; and $E^+(L) = \{xx\colon x \in W\} \cup \{xk\colon x \in S$ for some $S \subseteq U_1\}$.*
- *There exists a vertex $k \in W$ such that each extended triple in B containing k is a normal triple which contains at least one other vertex in W; and $E^+(L)$ consists of $\{xx\colon x \in W\}$ and the edges of a path of length three, with endvertices in U_1 and internal vertices k and b for some $b \in W \setminus \{k\}$.*
- *There exists a vertex $k \in U_1 \setminus U_0$ such that each extended triple in B containing k is either $\{k, k, k\}$ or a normal triple which contains at least one vertex in W; and $E^+(L)$ consists of $\{xx\colon x \in W\} \cup \{xk\colon x \in S$ for some $S \subseteq U_1\}$ and the edges of a path of length three, k, y, z, b, where $b \in U_1 \setminus \{k\}$, $y, z \in W$, and the third element in the extended triple containing the used edge kz is in W.*

Then (U_0, A_0) can be embedded in an extended triple system with underlying set $(U_1 \cup W) \setminus \{k\}$.

Proof. Consider the partial extended triple system $(U_1 \cup W, B) \setminus k$ and denote its leave by L'. Then $\{xy\colon \{k, x, y\} \in B\}$ is a matching in L'. It is straightforward to verify that L' consists of a union of vertex-disjoint trees, and a loop on each vertex of W. Furthermore, the conditions placed on the triples in B containing k guarantee that each tree contains at most one vertex of U_1. Thus we may apply Lemma 5.2 to obtain the required embedding of (U_0, A_0). □

Theorem 10.2. *Let $n \geqslant 9$. Any partial extended triple system of order n can be embedded in an extended triple system of order v if v is even and $v \geqslant 2n+4$.*

Proof. Let (U_0, A_0) be a partial extended triple system of order n, let v be even, $v \geqslant 2n+4$ and let L_0 be the leave of (U_0, A_0). Any such partial extended triple system can be embedded in a maximal partial extended triple system of order $\frac{v-4}{2}$ by arbitrarily adding extended triples. Hence we can assume that $v = 2n+4$ and (U_0, A_0) is maximal. By Theorem 5.6 we can assume L_0 is non-empty, and by Lemma 5.3 we can assume $\Delta(L_0) \leqslant n-2$. We split the proof into two cases:

- **Case 1:** $\beta((U_0, A_0)) \in \{n-2, n\}$.
- **Case 2:** $\beta((U_0, A_0)) \leqslant n-4$.

Case 1. If $\beta((U_0, A_0)) = n-2$ then by Lemma 5.4 we can assume $\epsilon(L) \geqslant 3$ and by Lemma 5.5 we can assume $L \neq K_{1,n-2} \cup K_1$. Thus Lemma 6.2 applies when $\beta((U_0, A_0)) = n-2$.

If $\beta((U_0, A_0)) = n$ we apply Lemma 6.1 to (U_0, A_0) and if $\beta((U_0, A_0)) = n-2$, we apply Lemma 6.2 to (U_0, A_0). In both cases we obtain an embedding of (U_0, A_0) in a partial extended triple system (U_1, A_1) of order $u = n+2$, with a useable leave L which contains no loops. Let N be the set of normal triples in A_1, let P be the set of proper extended triples in A_1, and let M denote the set of (non-loop) edges used in P.

Consider the partial Steiner triple system (U_1, N) of order $u = n+2$ and let its leave be L_1 (L_1 is a subgraph of K_{U_1}). Note that L_1 has edge set $E(L_1) = E(L) \cup M$, and L is a spanning useable subgraph of L_1. By Lemma 8.7 (which covers the case $\epsilon(L) \geqslant 3u-18$) and Lemma 9.9 (which covers the case $\epsilon(L) \leqslant 3u-21$), (U_1, N) can be embedded in a partial Steiner triple system $(U_1 \cup W, B)$ of order $2u+1 = 2n+5$ with a leave G such that either

- $E(G) = E(L_1) \setminus E(L) = M$; or
- $E(G)$ consists of $E(L_1) \setminus E(L) = M$ and the edges of a path of length three, with endvertices in U_1 and internal vertices in W.

Consider now the partial extended triple system $(U_1 \cup W, B \cup P)$ of order $2u+1 = 2n+5$ and denote its leave by L' (L' is a subgraph of $K^+_{U_1 \cup W}$). It is clear that $L'(U_1)$ is empty. From the remarks following Lemmas 8.7 and 9.9, it is also clear that one of the two following statements holds.

- There exists a vertex $a \in W$ such that each extended triple in $B \cup P$ containing a is a normal triple which contains at least one other vertex in W; and $E^+(L) = \{xx\colon x \in W\}$ (when $E(G) = M$).
- There exists a vertex $a \in W$ such that each extended triple in $B \cup P$ containing a is a normal triple which contains at least one other vertex in W; and $E^+(L)$ consists of $\{xx\colon x \in W\}$ and the edges of a path of length three, with endvertices in U_1 and internal vertices a and b for some $b \in W \setminus \{a\}$ (when $E(G)$ consists of M and the edges of a path of length three).

To obtain an embedding of (U_0, A_0) in an extended triple system of order $v = 2u = 2n + 4$, we apply Lemma 10.1 (with $k = a$) to $(U_1 \cup W, B \cup P)$.

Case 2. Let $p, s \notin U_0$, and let $U_1 = U_0 \cup \{p, s\}$. We apply Lemma 6.3 to embed (U_0, A_0) in a partial extended triple system (U_1, A_1) of order $u = n + 2$ with a leave L_1 which contains no loops, such that L_1 is the union of two edge-disjoint graphs L and S, where L is useable and every edge of S is incident with s. Let N be the set of normal triples in A_1, let P be the set of proper extended triples in A_1, and let M denote the set of (non-loop) edges used in P.

Consider the partial Steiner triple system (U_1, N) of order $u = n + 2$ and let its leave be L^* (L^* is a subgraph of K_{U_1}). Note that L^* has edge set $E(L^*) = E(L) \cup E(S) \cup M$, and L is a spanning useable subgraph of L^*. By Lemma 8.7 (which covers the case $\epsilon(L) \geqslant 3u - 18$) and Lemma 9.9 (which covers the case $\epsilon(L) \leqslant 3u - 21$), (U_1, N) can be embedded in a partial Steiner triple system $(U_1 \cup W, B)$ of order $2u + 1 = 2n + 5$ with a leave G such that either

- $E(G) = E(L^*) \setminus E(L) = E(S) \cup M$; or
- $E(G)$ consists of $E(S) \cup M$ and the edges of a path of length three, with endvertices in U_1 and internal vertices in W.

From the remarks following Lemmas 6.3, 8.7 and 9.9, it is clear that if $E(G)$ consists of $E(S) \cup M$ and the edges of a path of length three, then s is an endvertex of the path.

Consider now the partial extended triple system $(U_1 \cup W, B \cup P)$ of order $2u + 1 = 2n + 5$ and denote its leave by L' (L' is a subgraph of $K^+_{U_1 \cup W}$). It is clear that $E^+(L'(U_1)) = E(S)$. The constructions in Lemmas 6.3, 8.7 and 9.9 ensure that each extended triple in $B \cup P$ containing s is either $\{s, s, s\}$ or a normal triple which contains at least one vertex in W. Furthermore, it is clear that one of the following two statements holds.

- $E^+(L') = \{xx\colon x \in W\} \cup E(S)$ (when $E(G) = E(S) \cup M$).
- $E^+(L')$ consists of $\{xx\colon x \in W\} \cup E(S)$ and the edges of a path of length three, s, y, z, b, where $b \in U_1 \setminus \{s\}$, $y, z \in W$, and the third element in the extended triple containing the used edge sz is in W (when $E(G)$ consists of $E(S) \cup M$ and the edges of a path of length three).

To obtain an embedding of (U_0, A_0) in an extended triple system of order $v = 2u = 2n + 4$, we apply Lemma 10.1 (with $k = s$) to $(U_1 \cup W, B \cup P)$. $\square$

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References

[1] L.D. Andersen, A.J.W. Hilton, E. Mendelsohn, Embedding partial Steiner triple systems, Proc. London Math. Soc. 41 (1980) 557–576.
[2] F. Bennett, Quasigroups, in: C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, CRC Press, 1996.
[3] D. Bryant, D. Horsley, A proof of Lindner's conjecture on embeddings of partial Steiner triple systems, preprint.
[4] D. Bryant, A. Khodkar, Maximum packings of $K_v - K_u$ with triples, Ars Combin. 55 (2000) 259–270.
[5] D. Bryant, Embeddings of partial Steiner triple systems, J. Combin. Theory Ser. A 106 (2004) 77–108.

[6] R. Chaffer, M. Eggen, R. St. Andre, D. Smith, Strong finite embeddability for classes of quasigroups, Algebra Universalis 5 (1975) 257–262.
[7] A.J.W. Hilton, C.A. Rodger, Graphs and configurations, in: Regards sur la théorie des graphes, Proc. Colloq., Cerisy, 1980, Presses Polytech. Romandes, Lausanne, 1980, pp. 55–64.
[8] D.G. Hoffman, C.A. Rodger, Embedding totally symmetric quasigroups, in: Ann. Discrete Math., vol. 34, 1987, pp. 249–258.
[9] D. Johnson, N.S. Mendelsohn, Extended triple systems, Aequationes Math. 8 (1972) 291–298.
[10] C.C. Lindner, A.B. Cruse, Small embeddings for partial semisymmetric and totally symmetric quasigroups, J. London Math. Soc. 12 (1976) 479–484.
[11] C.J.H. McDiarmid, The solution of a timetabling problem, J. Inst. Math. Appl. 9 (1972) 23–34.
[12] M.E. Raines, More on embedding partial totally symmetric quasigroups, Australas. J. Combin. 14 (1996) 297–309.
[13] M.E. Raines, C.A. Rodger, Embedding partial extended triple systems and totally symmetric quasigroups, Discrete Math. 176 (1997) 211–222.
[14] J. Schönheim, On maximal systems of k-tuples, Studia Sci. Math. Hungar. 1 (1966) 363–368.
[15] P. Turán, An extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452.
[16] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz 3 (1964) 25–30 (in Russian).
[17] D. de Werra, Equitable colorations of graphs, Rev. Fran. Inf. Rech. Opr. 5 (1971) 3–8.